

Components, Coalgebras, and Chu Spaces

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- 1 Motivations
- 2 Components as coalgebras
 - Coalgebras
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- 3 Component and Chu Spaces

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Two questions re Software Components in PiCoq

① Semantics ?

② Calculus ?

- Formal component metamodel
- Various form of behavioral composition
- Sharing & aspects
- Extensions : stochastic & hybrid models

- Process calculus for component constructs
- Universality wrt component metamodel
 - *Any r.e. component can be realized*
- Coalgebraic semantics

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- Streams
- Infinite data structures
- Labelled transition systems
- Self-applicative & reflective programs

- $x : X \times X \rightarrow X$
 $y : X \times X \times X \rightarrow X$
 $z : 1 \rightarrow X$
- $f : \Sigma(X) \rightarrow X$
- homomorphism: from $f : \Sigma(A) \rightarrow A$ to $g : \Sigma(B) \rightarrow B$
 $h : A \rightarrow B$ extends to $\Sigma(h) : \Sigma(A) \rightarrow \Sigma(B)$
- initial Σ -algebra: $id_{\Sigma_*} : \Sigma(\Sigma_*) \rightarrow \Sigma_*$
there exists a unique homomorphism
 h_f from $id_{\Sigma_*} : \Sigma(\Sigma_*) \rightarrow \Sigma_*$ to $f : \Sigma(X) \rightarrow X$
- Σ_* is the least fixed point of operator Σ
(Σ conceived as operator on sets)

- $f : X \rightarrow \Gamma(X)$
- homomorphism: from $f : A \rightarrow \Gamma(A)$ to $g : B \rightarrow \Gamma(B)$
 $h : A \rightarrow B$ extends to $\Gamma(h) : \Gamma(A) \rightarrow \Gamma(B)$
- final Γ -coalgebra (Γ monotone): $id_{\Gamma^*} : \Gamma^* \rightarrow \Gamma(\Gamma^*)$
there exists a unique homomorphism
 h_f from $f : A \rightarrow \Gamma(A)$ to $id_{\Gamma^*} : \Gamma^* \rightarrow \Gamma(\Gamma^*)$
- Γ^* is the greatest fixed point of operator Γ

- Simple set-based models: “concrete” and “intuitive”
- Working up-to bisimulation equivalence factored-in
- Greatest non-trivial fixed-points for key operators

What's an hyperset ?

- A non-well-founded set
 - $\Omega = \{\Omega\}$
 - $s = \langle a, s \rangle$
 - $x = \{\langle a, y \rangle, \langle b, x \rangle\}$
 $y = \{\langle c, z \rangle, \langle a, x \rangle\}$
 $z = \{\langle c, z \rangle, \langle b, x \rangle\}$
- The unique solution of a flat system of equation (Anti-Foundation Axiom)
 - A flat system of equation is a function: $e : X \rightarrow \mathcal{P}(X \cup A)$
 X a set of variables
 A a set disjoint from X
- A pointed coalgebra of the functor $\mathcal{P}(X \cup A)$

Fixed points of functors

- Functor: $\Gamma : \mathcal{U}_{afa} \rightarrow \mathcal{U}_{afa}$
 \mathcal{U}_{afa} class of all (hyper)sets
- Monotone functor: $a \subseteq b \implies \Gamma(a) \subseteq \Gamma(b)$
- G fixed point of Γ is $G = \Gamma(G)$
- Least fixed point Γ_* : for any G fixed point $\Gamma_* \subseteq G$
 - **Induction principle**: to prove $\Gamma_* \subseteq G$, show that $\Gamma(G \cap \Gamma_*) \subseteq G$
- Greatest fixed point Γ^* : for any G fixed point $G \subseteq \Gamma^*$
 - **Coinduction principle**: to prove $G \subseteq \Gamma^*$, show that $G \subseteq \Gamma(G \cup \Gamma^*)$
- Every monotone functor Γ has a least and a greatest fixed point

Fixed points of functors

$\Gamma(a)$	Γ_*	Γ^* (with FA)	Γ^* (with AFA)
a	\emptyset	\mathcal{U}_{wf}	\mathcal{U}_{afa}
$\mathcal{P}(a)$	\mathcal{U}_{wf}	\mathcal{U}_{wf}	\mathcal{U}_{afa}
$A \times a$	\emptyset	\emptyset	infinite streams over A
$A \times a \times a$	\emptyset	\emptyset	infinite binary trees over A
$\mathcal{P}(A \times a)$	\emptyset	\emptyset	canonical LTS over A

Theorem (Representation theorem for greatest fixed points)

If Γ is uniform (i.e. monotone + some other conditions)

$$\Gamma^* = \{ \text{solution}(\mathcal{E}) \mid \mathcal{E} \text{ a } \Gamma\text{-coalgebra} \}$$

Definition

Let Γ be a monotone functor. A Γ -invariant is a predicate P on Γ^* , i.e. a subset of Γ^* , such that if $P(c)$ holds then $P(k)$ holds for all $k \in TC(c) \cap \Gamma^*$.

Theorem

Let Γ be a uniform functor, and let P be a predicate on Γ^ . Let P^* be the greatest Γ -invariant contained in P . Then $\langle P^*, id_{P^*} \rangle$ is a Γ -coalgebra.*

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Definition (The Fractal functor)

$$\Gamma(X) = \mathcal{M}_f(X) \times \mathcal{P}(\mathcal{M}_f(\Sigma(X)) \times \mathcal{M}_f(\Sigma(X)) \times \mathcal{M}_f(X))$$

- $\Sigma(X) = \coprod_{k \in \mathbb{N}} (\mathbb{L} \times \Delta(X))^k$
 - $\Delta(X) = \mathbb{L} + \mathbb{V} + X$
 - \mathbb{L} set of **names**
 - \mathbb{V} set of **values**
 - $\mathcal{M}_f(S)$ finite multisets of elements from S
 - $\mathcal{P}(S)$ subsets of S
-
- $\Gamma^* = \Gamma(\Gamma^*)$ set of **components**
 - $\Sigma(\Gamma^*)$ set of **signals**
 - $\Delta(\Gamma^*)$ set of **arguments**

Fractal components as pointed coalgebras

- Fractal components are solutions to pointed equations: $e_x : X \rightarrow \Gamma(X)$

- Example:

$$x = \{y_1, y_2\} \cdot \{\langle\{s_1, s_2\}, \{r\}, y\rangle, \langle\{\text{ping}\}, \{\text{ok}\}, x\rangle\}$$

$$y = \{y_1\} \cdot \{\langle\{s_3\}, \{\text{ok}\}, x\rangle\}$$

$$s_1 = \langle\text{op} : \text{lock}\rangle$$

$$s_2 = \langle\text{op} : \text{get}, \text{arg} : \text{snd}\rangle$$

$$r = \langle\text{return} : y_2\rangle$$

$$s_3 = \langle\text{op} : \text{insert}, \text{arg} : y_2\rangle$$

- in **component** $x = \{y_1, y_2\} \cdot \{t_1, t_2\}$:

- y_1, y_2 are x 's **sub-components**
- t_1, t_2 are x 's **transitions**

- in **transition** $t_1 = \langle\{s_1, s_2\}, \{r\}, y\rangle$:

- y is t_1 's **residue**
- s_1, s_2 are t_1 's **input signals**
- r is t_1 's **output signal**

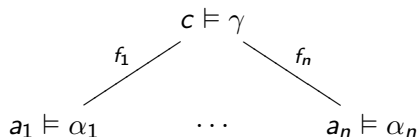
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The nature of composition

- Composition as algebraic operator:

$$op : \Sigma(X) \rightarrow X$$

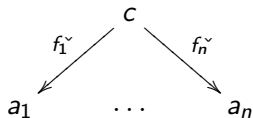
- Composition and types:



- Chu space $\mathbf{A} = (A, \Sigma, \vDash)$
 - A tokens (components), Σ types
 - $a \vDash \alpha$: token a is of type α

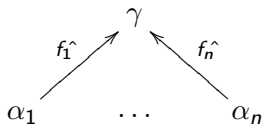
The nature of composition: effect on tokens

Composition identifies tokens in a composite:



The nature of composition: effect on types

Composition combines types in a composite:



Composition as Chu-morphism

$$\begin{array}{ccc} \Sigma_A & \xrightarrow{f^\wedge} & \Sigma_C \\ \vDash_A \Big| & & \Big| \vDash_C \\ A & \xleftarrow{f^\vee} & C \end{array}$$

$$c \vDash f_i^\wedge(\alpha_i) \iff f_i^\vee(c) \vDash \alpha_i$$

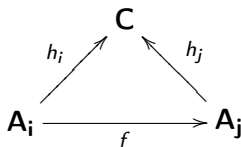
Systems as colimits of Chu spaces

Model an assemblage \mathbf{A} as a collection of Chu spaces and Chu morphisms

$$\{\mathbf{A}_i\}_{i \in I} \cup \{f_k : \mathbf{A}_{i_k} \rightarrow \mathbf{A}_{j_k}\}_{k \in K}$$

Theorem

Every composition system has a colimit and it is unique up to isomorphism.



- How to reconcile the two views ?
 - Components as pointed coalgebras
 - Composition as Chu-morphism
 - Cf Abramsky: category of Chu spaces as full subcategory of Grothendiek category of coalgebras
- Can we find a universal component functor ?
 - Every composition can be realized
- Can we find a universal language for the component functor ?
 - Every r.e. pointed coalgebra can be realized

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