The probabilistic modal $\mu$-calculus with independent product

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The Modal $\mu$-Calculus: $L\mu$

- Interpreted on $LTS = \langle P, \{ a \rightarrow \} \rangle_{a \in L}$, with $a \rightarrow \subseteq P \times P$.
- Syntax: $F ::= X | F \lor G | F \land G | \langle a \rangle F | [a] F | \mu X.F | \nu X.F$
- Semantics: $\llbracket F \rrbracket_\rho \subseteq P$, with $\rho : Var \rightarrow 2^P$
  \[
  \llbracket F \rrbracket_\rho = \rho(X) \\
  \llbracket F \lor G \rrbracket_\rho = \llbracket F \rrbracket_\rho \cup \llbracket G \rrbracket_\rho \\
  \llbracket F \land G \rrbracket_\rho = \llbracket F \rrbracket_\rho \cap \llbracket G \rrbracket_\rho \\
  \llbracket \langle a \rangle F \rrbracket_\rho = \{ p \mid \exists q.p \xrightarrow{a} q, q \in \llbracket F \rrbracket_\rho \} \\
  \llbracket [a] F \rrbracket_\rho = \{ p \mid \forall q.p \xrightarrow{a} q \text{ implies } q \in \llbracket F \rrbracket_\rho \} \\
  \llbracket \mu X.F \rrbracket_\rho = \text{lfp of } \lambda S. \llbracket F \rrbracket_\rho[S/X] \\
  \llbracket \nu X.F \rrbracket_\rho = \text{gfp of } \lambda S. \llbracket F \rrbracket_\rho[S/X]
\]
The Modal $\mu$-Calculus: $L\mu$

- Interpreted on $\text{LTS} = \langle P, \{ \stackrel{a}{\to}\}_{a \in L} \rangle$, with $\stackrel{a}{\to} \subseteq P \times P$.
- Syntax: $F ::= X | F \lor G | F \land G | \langle a \rangle F | [a] F | \mu X.F | \nu X.F$
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\[
\begin{align*}
\llbracket F \rrbracket_\rho &= \rho(X) \\
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\llbracket F \land G \rrbracket_\rho &= \llbracket F \rrbracket_\rho \cap \llbracket G \rrbracket_\rho \\
\llbracket \langle a \rangle F \rrbracket_\rho &= \{ p | \exists q. p \stackrel{a}{\to} q, q \in \llbracket F \rrbracket_\rho \} \\
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\end{align*}
\]

Expressivity: bisimilarity-invariant fragment of $\text{MSO}$ (Janin, Walukiewicz 1996).
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- Syntax: $F ::= X \mid F \land G \mid F \lor G \mid \langle a \rangle F \mid [a] F \mid \mu X.F \mid \nu X.F$
- Semantics: $\llbracket F \rrbracket_\rho : P \rightarrow \{0, 1\}$, with $\rho : \text{Var} \rightarrow (P \rightarrow \{0, 1\})$

\[
\begin{align*}
\llbracket F \lor G \rrbracket = \llbracket F \rrbracket \uplus \llbracket G \rrbracket \\
\llbracket \langle a \rangle F \rrbracket(p) = \bigsqcup_{p \xrightarrow{a} q} \llbracket F \rrbracket(q) \\
\llbracket [a] F \rrbracket(p) = \bigsqcap_{p \xrightarrow{a} q} \llbracket F \rrbracket(q) \\
\llbracket \mu X.F \rrbracket = \text{lfp of } \lambda f. \llbracket F \rrbracket_{\rho[f/X]} \\
\llbracket \nu X.F \rrbracket = \text{gfp of } \lambda f. \llbracket F \rrbracket_{\rho[f/X]} 
\end{align*}
\]
Game Semantics: a few examples

\[ F = \nu X. \langle a \rangle X \]
There exist a infinite \( a \)-path.

\[ G = \mu X. [a] X \]
Every \( a \)-path is finite.

\[ H = \langle a \rangle [a] \perp \]
There is some \( a \)-step, after which no further \( a \)-step is possible.
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Player ♦ either gets stuck, or can force the play into an infinite \( \nu \)-loop.

So, ♦ wins this game: \( \llbracket F \rrbracket (p) = 1 \).
Game Semantics: a few examples

$a \xrightarrow{a} p \xrightarrow{a} q$

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There exist a infinite $a$-path.

$G = \mu X. [a] X$
Every $a$-path is finite.

$H = \langle a \rangle [a] \perp$
There is some $a$-step, after which no further $a$-step is possible.

Game associated with $G$:

Player $\square$ either gets stuck, or can force the play into an infinite $\mu$-loop.

So, $\square$ wins this game, i.e. $\Diamond$ loses: $
\llbracket G \rrbracket (p) = 0.$
Game Semantics: a few examples

\[ \text{Game associated with } H: \]

\[ F = \nu X . \langle a \rangle X \]

There exist an infinite \( a \)-path.

\[ G = \mu X . [a] X \]

Every \( a \)-path is finite.

\[ H = \langle a \rangle [a] \perp \]

There is some \( a \)-step, after which no further \( a \)-step is possible.

Player ♦ can make sure Player □ will get stuck.

So, ♦ wins this game: \([H](p) = 1\).
Theorem [e.g. Stirling 96]:
\( [F](p) = 1 \) iff ♦ has a winning strategy in \( G^F \) from \( (p:F) \).

- Denotational Semantics and Game Semantics coincide.
- Useful to have an operational interpretation for the meaning of a formula.
- Game Semantics very successful: theoretical results, model checking algorithms, ...
A PLTS is a pair \( \langle P, \{ a \rightarrow \} \rangle \) where

- \( P \) is a countable set of states,
- \( L \) is a countable set of labels, or atomic actions,
- \( a \rightarrow \subseteq P \times \mathcal{D}(P) \) is the \( a \)-transition relation.
A PLTS is a pair $\langle P, \{a \rightarrow\}_a \in L \rangle$ where

- $P$ is a countable set of states,
- $L$ is a countable set of labels, or atomic actions,
- $a \rightarrow \subseteq P \times \mathcal{D}(P)$ is the $a$-transition relation.
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- Semantics: $\llbracket F \rrbracket_\rho : P \rightarrow \{0, 1\}$, with $\rho : Var \rightarrow (P \rightarrow \{0, 1\})$

\[
\begin{align*}
\llbracket F \lor G \rrbracket &= \llbracket F \rrbracket \cup \llbracket G \rrbracket \\
\llbracket \langle a \rangle F \rrbracket(p) &= \bigcup_{p \xrightarrow{a} q} \llbracket F \rrbracket(q) \\
\llbracket \mu X.F \rrbracket &= \text{lfp of } \lambda f. \llbracket F \rrbracket_\rho[f/X] \\
\llbracket \nu X.F \rrbracket &= \text{gfp of } \lambda f. \llbracket F \rrbracket_\rho[f/X]
\end{align*}
\]
The Probabilistic Modal $\mu$-Calculus: $pL\mu$


- Interpreted on $\text{PLTS} = \langle P, \{ \stackrel{a}{\rightarrow} \}_{a \in L} \rangle$, with $\stackrel{a}{\rightarrow} \subseteq P \times \mathcal{D}(P)$.

- Syntax: $F ::= X \mid F \lor G \mid F \land G \mid \langle a \rangle F \mid [a] F \mid \mu X.F \mid \nu X.F$

- Semantics: $\llbracket F \rrbracket_\rho : P \rightarrow [0, 1]$, with $\rho : \text{Var} \rightarrow (P \rightarrow [0, 1])$

$$\begin{align*}
\llbracket F \lor G \rrbracket & = \llbracket F \rrbracket \sqcup \llbracket G \rrbracket \\
\llbracket \langle a \rangle F \rrbracket(p) & = \bigsqcup_{p \stackrel{a}{\rightarrow} \alpha} \llbracket F \rrbracket(\alpha) \\
\llbracket \mu X.F \rrbracket & = \text{lfp of } \lambda f. \llbracket F \rrbracket_{\rho[f/X]} \\
\llbracket [a] F \rrbracket(p) & = \bigsqcap_{p \stackrel{a}{\rightarrow} \alpha} \llbracket F \rrbracket(\alpha) \\
\llbracket \nu X.F \rrbracket & = \text{gfp of } \lambda f. \llbracket F \rrbracket_{\rho[f/X]} 
\end{align*}$$
The Probabilistic Modal $\mu$-Calculus: $pL\mu$

- Interpreted on $PLTS = \langle P, \{ \stackrel{\alpha}{\rightarrow} \}_{a \in L} \rangle$, with $\stackrel{\alpha}{\rightarrow} \subseteq P \times D(P)$.
- Syntax: $F ::= X | F \lor G | F \land G | \langle a \rangle F | [a] F | \mu X.F | \nu X.F$
- Semantics: $[F]_\rho : P \rightarrow [0, 1]$, with $\rho : Var \rightarrow (P \rightarrow [0, 1])$

\[
[F \lor G] = [F] \sqcup [G] \quad [F \land G] = [F] \sqcap [G]
\]
\[
[\langle a \rangle F](p) = \bigcup_{p \stackrel{\alpha}{\rightarrow} \alpha} [F](\alpha)
\]
\[
[\mu X.F] = \text{lfp of } \lambda f \cdot [F]_{\rho[f/X]}
\]
\[
[\nu X.F] = \text{gfp of } \lambda f \cdot [F]_{\rho[f/X]}
\]
\[
[F](\alpha) = \sum_{p \in P} \alpha(p) \cdot [F](p)
\]
Game Semantics for $\mathsf{pL}\mu$: a few examples

\[
F = \nu X. \langle a \rangle X
\]
There exist an infinite $a$-path.

\[
G = \mu X. [a] X
\]
Every $a$-path is finite.

\[
H = \langle a \rangle [a] \perp
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There is some $a$-step, after which no further $a$-step is possible.
Game Semantics for $\text{pL}_\mu$: a few examples

Game associated with $F$:

$$F = \nu X. \langle a \rangle X$$

There exist a infinite $a$-path.

$$G = \mu X. [a] X$$

Every $a$-path is finite.

$$H = \langle a \rangle [a] \bot$$

There is some $a$-step, after which no further $a$-step is possible.

Player ♦ either gets stuck (and lose), or end up in an infinite $\nu$-loop (and win).

However, the probability of winning is 0.

So, ♦ wins this game with probability 0: $\llbracket F \rrbracket (p) = 0$. 
Game Semantics for pLμ: a few examples

\[ F = \nu X.\langle a\rangle X \]
There exist a infinite \( a \)-path.

\[ G = \mu X.\lbrack a\rbrack X \]
Every \( a \)-path is finite.

\[ H = \langle a\rangle \lbrack a\rbrack \perp \]
There is some \( a \)-step, after which no further \( a \)-step is possible.

Game associated with \( G \):

Player \( \square \) either gets stuck (and lose), or end up in a infinite \( \mu \)-loop (and win).

\( \text{However, this happens with prob. 0.} \)

So, \( \lozenge \) wins this game with probability 1: \( \llbracket G \rrbracket (p) = 1. \)
Game Semantics for $pL\mu$: a few examples

There exist a infinite $a$-path.

Every $a$-path is finite.

There is some $a$-step, after which no further $a$-step is possible.

Player $\Diamond$ reaches $\bot$ with prob. $\frac{1}{3}$, and $\Box$ gets stuck with probability $\frac{2}{3}$.

So, $\Diamond$ wins this game with probability $\frac{2}{3}$: $\llbracket H \rrbracket (p) = \frac{2}{3}$. 

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There exist a infinite $a$-path.

Every $a$-path is finite.

There is some $a$-step, after which no further $a$-step is possible.

**In general**

Best probability of satisfying $F$ (read as in $L_\mu$) against any hostile environment.
Theorem [Mio 2010, Morgan and McIver 2004]:

$$\llbracket F \rrbracket (p) = \text{value of the game } G^F \text{ at } (p:F).$$

where the (quantitative) value is defined as usual in game theory:

$$\bigcup \bigcap_{\sigma^\Diamond} E(M_{\sigma^\Diamond, \sigma^\Box}) = \bigcap \bigcup_{\sigma^\Box} E(M_{\sigma^\Diamond, \sigma^\Box})$$

- Denotational Semantics and Game Semantics coincide.
- Useful to have an operational interpretation for the meaning of a probabilistic sentences.
- Game Semantics provides: model checking algorithms, ...
Interpreted on PLTS = \( \langle P, \{ \xrightarrow{a} \}_{a \in L} \rangle \), with \( \xrightarrow{a} \subseteq P \times \mathcal{D}(P) \).

Syntax: \[ F ::= X \mid F \lor G \mid F \land G \mid \langle a \rangle F \mid [a] F \mid \mu X.F \mid \nu X.F \mid F \otimes G \mid F \cdot G \]

Semantics: \[ [F]_{\rho} : P \rightarrow [0, 1], \text{ with } \rho : \text{Var} \rightarrow (P \rightarrow [0, 1]) \]
\[ [F \cdot G] (p) = [F] (p) \cdot [G] (p) \quad [F \otimes G] (p) = [F] (p) \odot [G] (p) \]
where \( x \odot y = x + y - (x \cdot y) \)

- the De Morgan dual of \( \cdot \) under \( \neg x = 1 - x \): \( x \odot y \overset{\text{def}}{=} \neg(\neg x \cdot \neg y) \).
Interpreted on PLTS = \langle P, \{ \overset{a}{\rightarrow}\}_{a \in L} \rangle, with \overset{a}{\rightarrow} \subseteq P \times D(P).

Syntax: \( F ::= X \mid F \lor G \mid F \land G \mid \langle a \rangle F \mid [a] F \mid \mu X.F \mid \nu X.F \)
\( F \circ G \mid F \cdot G \)

Semantics: \([F]_\rho : P \rightarrow [0, 1], \text{ with } \rho : \text{Var} \rightarrow (P \rightarrow [0, 1])\)
\([F \cdot G](p) = [F](p) \cdot [G](p) \quad [F \circ G](p) = [F](p) \circ [G](p)\)

where \( x \circ y = x + y - (x \cdot y)\)

- the De Morgan dual of \( \cdot \) under \( \neg x = 1 - x: x \circ y \overset{\text{def}}{=} \neg(\neg x \cdot \neg y).\)

Mathematically well defined (\( \cdot \) and \( \circ \) are monotone).
Interpreted on PLTS = $\langle P, \{\xrightarrow{a}\}_{a \in L} \rangle$, with $\xrightarrow{a} \subseteq P \times D(P)$.

Syntax: $F ::= X \mid F \lor G \mid F \land G \mid \langle a \rangle F \mid [a] F \mid \mu X.F \mid \nu X.F \mid F \odot G \mid F \cdot G$

Semantics: $\llbracket F \rrbracket_\rho : P \rightarrow [0, 1]$, with $\rho : Var \rightarrow (P \rightarrow [0, 1])$

$\llbracket F \cdot G \rrbracket (p) = \llbracket F \rrbracket (p) \cdot \llbracket G \rrbracket (p)$
$\llbracket F \odot G \rrbracket (p) = \llbracket F \rrbracket (p) \circ \llbracket G \rrbracket (p)$

where $x \circ y = x + y - (x \cdot y)$

the De Morgan dual of $\cdot$ under $\neg x = 1 - x$: $x \odot y \overset{\text{def}}{=} \neg(\neg x \cdot \neg y)$.

Mathematically well defined ($\cdot$ and $\odot$ are monotone).

But is it meaningful?

$\llbracket F \cdot G \rrbracket$ probability that $F$ and $G$ holds independently?

$\llbracket F \odot G \rrbracket$ probability that $F$ or $G$ holds independently?
Why $pL\mu^{\circ}$??

Let us define

- $\mathbb{P}_{>0} F \overset{\text{def}}{=} \mu X.(F \odot X)$, and
- $\mathbb{P}_{=1} F \overset{\text{def}}{=} \nu X.(F \cdot X)$.

Then

- $\llbracket \mathbb{P}_{>0} F \rrbracket (p) = \begin{cases} 1 & \text{if } \llbracket F \rrbracket (p) > 0 \\ 0 & \text{otherwise} \end{cases}$

- $\llbracket \mathbb{P}_{=1} F \rrbracket (p) = \begin{cases} 1 & \text{if } \llbracket F \rrbracket (p) = 1 \\ 0 & \text{otherwise} \end{cases}$

This allows:

- the expression of interesting (new) properties involving qualitative/quantitative assertions (see paper).
- The encoding of the qualitative fragment of PCTL into $pL\mu^{\circ}$. 
Game Semantics for pLμ: a few examples

Game associated with $H \cdot J$:

$$H = \langle a \rangle [a] \bot$$
Probability to reach after some $a$-step
a state without $a$-edges.

$$J = \langle a \rangle \langle a \rangle ^{\top}$$
Probability to reach after some $a$-step
a state with some $a$-edge.

$$H \cdot J$$
Probability of satisfying both $H$ and $J$
when $H$ and $J$ independently verified.
Game Semantics for pL₂: a few examples

\[ H = \langle a \rangle [a] \perp \]
Probability to reach after some a-step a state without a-edges.

\[ J = \langle a \rangle \langle a \rangle \top \]
Probability to reach after some a-step a state with some a-edge.

\[ H \cdot J \]
Probability of satisfying both \( H \) and \( J \) when \( H \) and \( J \) independently verified.

\[ \langle H \cdot J \rangle (p) = \frac{2}{9}. \]

Game associated with \( H \cdot J \):

\[ \diamond \text{ wins iff } \text{He wins in both sub-games.} \]

\[ \text{Since they are independent, this will happen with probability } \frac{1}{3} \cdot \frac{2}{3}. \]
Game Semantics for pL\(\mu\): a few examples

Game associated with \(\mu X.(H \odot X)\):

\[
H = \langle a \rangle [a] \perp
\]
Probability to reach after some \(a\)-step a state without \(a\)-edges.

\[
\mathbb{P}_{>0} H = \mu X.(X \odot H)
\]
1 if \(H\) is possible,
0 otherwise.

\[
\uparrow
\]
probability that \(H\) holds at least once if verified infinitely many times.
Game Semantics for pLµ: a few examples

\[ H = \langle a \rangle [a] \perp \]

Probability to reach after some a-step a state without a-edges.

\[ \mathbb{P}_{>0} H = \mu X. (X \odot H) \]

1 if \( H \) is possible,
0 otherwise.

\[ \uparrow \]

probability that \( H \) holds at least once if verified infinitely many times.

\[ \Diamond \text{ will win in at least on sub-game almost surely!} \]

\[ \llbracket \mu X. (H \odot X) \rrbracket (p) = 1. \]
These ideas are formalized using $2\frac{1}{2}$-player tree games, which build on the intuitive idea of concurrent and independent execution of sub-games.

A new class of games having trees as outcomes, rather than paths.

The branches of the trees are generated when the game is split in concurrent and independent sub-games.

The winning-set of a $2\frac{1}{2}$-player tree game is a set $\Phi$ of trees which we call branching plays.

In the case of $pL\mu\diamond$ games the winning set, is the set of trees such that $\diamond$ can find a winning path by making or choices at the branching nodes $p : F \diamond G$, against any and choice made by $\boxdot$ on the nodes $p : F \cdot G$.

i.e. the trees that, once interpreted as ordinary 2-player parity games, are won by $\diamond$.

That’s why we call them $2\frac{1}{2}$-player meta-parity games.
One can define the notion of (upper and lower) \textbf{value} of a $2\frac{1}{2}$-player tree game.

\[ Val_{\downarrow}(G) = \bigcup \bigcap E_{\sigma^\Diamond,\sigma^\Box}(\Phi) \]

\[ Val_{\uparrow}(G) = \bigcap \bigcup E_{\sigma^\Box,\sigma^\Diamond}(\Phi) \]

\textbf{Theorem (MA$_{\aleph_1}$):} If $G$ is a pL$\mu^\Diamond$ game, then:

\[ Val_{\downarrow}(G) = Val_{\uparrow}(G). \]

\textbf{Theorem (MA$_{\aleph_1}$):} For every pL$\mu^\Diamond$ formula $F$:

\[ [F](p) = \text{value of } G^F \text{ at } (p, F). \]
Our theorems hold in $\text{ZFC} + \text{MA}_{\aleph_1}$ set theory.

- $\text{MA}$ is an axiom considered by set theorists as a weaker alternative to $\text{CH}$.

- $\text{MA}_{\aleph_1}$ is a consequence of $\text{MA} + \neg\text{CH}$ and itself implies $\neg\text{CH}$.

- In particular it implies that:
  - measurable sets are closer under $\omega_1$ unions.
  - measures are $\omega_1$-continuous.

- Therefore our proof is a consistent proof.
Our theorems hold in $\text{ZFC} + \text{MA}_{\mathbb{N}_1}$ set theory.

- $\text{MA}$ is an axiom considered by set theorists as a weaker alternative to $\text{CH}$.
- $\text{MA}_{\mathbb{N}_1}$ is a consequence of $\text{MA} + \neg \text{CH}$ and itself implies $\neg \text{CH}$.
- In particular it implies that:
  - measurable sets are closer under $\omega_1$ unions.
  - measures are $\omega_1$-continuous.

- Therefore our proof is a consistent proof.

- Also Fermat’s Last Theorem is proved in $\text{ZFC} + \text{U!!}$
  \[ \forall a, b, c \in \mathbb{Z}. a^n + b^n \neq c^n, \text{ when } n > 3. \]
We use $\mathsf{MA}_{\aleph_1}$ to handle the complexity of the winning sets $\Phi$ of $\mu^\circ$ games.

► We prove that $\Phi$ is always a $\Delta^1_2$ set.

► Hence not Borel, and not necessarily measurable.

► But we characterize $\Phi$ as a $\omega_1$-union of measurable sets: $\Phi = \bigcup_{\alpha < \omega_1} \Phi^\alpha$.

► Hence, under $\mathsf{MA}_{\aleph_1}$, $\Phi$ is measurable, and its measure is the limit of the measures of the approximants.

$$\mu(\Phi) = \bigsqcup_{\alpha < \omega_1} \mu(\Phi^\alpha)$$
Many open problems!
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- can $\text{MA}_{\aleph_1}$ be dropped from the proof?
Many open problems!

- Can $\text{MA}_{\aleph_1}$ be dropped from the proof?
- Is a finite $pL_{\mu^{\otimes}}$-game positionally determined?
Many open problems!

- Can $\text{MA}_{\aleph_1}$ be dropped from the proof?
- Is a finite $\text{pL}_{\mu^\square}$-game positionally determined?
  - Conjecture was: YES!
Many open problems!

- Can $\text{MA}_{\aleph_1}$ be dropped from the proof?
- Is a finite $\text{pL}_\mu^\omega$-game positionally determined?
  - Conjecture was: YES!
  - Answer is: NO!
Many open problems!

- can $\text{MA}_{\aleph_1}$ be dropped from the proof?
- Is a finite $\text{pL}_{\mu^\ominus}$-game positionally determined?
  - Conjecture was: YES!
  - Answer is: NO!
- Is the value of a finite $\text{pL}_{\mu^\ominus}$-game decidable? !!!
Many open problems!

- Can $\text{MA}_{\aleph_1}$ be dropped from the proof?
- Is a finite $\text{pL}_\mu^\odot$-game positionally determined?
  - Conjecture was: YES!
  - Answer is: NO!
- Is the value of a finite $\text{pL}_\mu^\odot$-game decidable? !!!
  - Failure of positional determinacy makes this problem challenging.
Many open problems!

- Can $\text{MA}_{\aleph_1}$ be dropped from the proof?
- Is a finite $pL\mu\circledast$-game positionally determined?
  - Conjecture was: YES!
  - Answer is: NO!
- Is the value of a finite $pL\mu\circledast$-game decidable? !!!
  - Failure of positional determinacy makes this problem challenging.
- Study the logical-equivalence (or metric) induced by the logic $pL\mu\circledast$, or even the modal fragment $\{\top, \bot, \lor, \land, \langle a \rangle, [a], \cdot, \otimes\}$.
THANKS
A few interesting examples

**Figure:** Example of PLTS

1. $F_2 \overset{\text{def}}{=} \nu X. \langle a \rangle X$
   
   “Best probability of making an infinite sequence of a’s”.
A few interesting examples

1. $F_2 \overset{\text{def}}{=} \nu X \cdot \langle a \rangle X$
   “Best probability of making an infinite sequence of $a$’s”.

2. $F_3 \overset{\text{def}}{=} \mu X \cdot (F_2 \lor \langle b \rangle X)$
   “Best probability of making a finite sequence of $b$’s followed by an infinite sequence of $a$’s”.

Figure: Example of PLTS
A few interesting examples

\[ F_2 \overset{\text{def}}{=} \nu X.\langle a \rangle X \]
“Best probability of making an infinite sequence of a’s”.

\[ F_3 \overset{\text{def}}{=} \mu X.\left( F_2 \lor \langle b \rangle X \right) \]
“Best probability of making a finite sequence of b’s followed by an infinite sequence of a’s”.

\[ F_5 \overset{\text{def}}{=} \langle a \rangle \langle a \rangle 1 \land [a] [a] 0 \]
\[ 0 \leq \lbrack F_5 \rbrack (p) \leq \frac{1}{2} \text{ for all } p \]
The logic is not Boolean! (Kleene Algebra)
Figure: Example of PLTS

1. \( G_1 \overset{\text{def}}{=} P_{=1}(\nu X.\langle a\rangle X) \)
   
   “Holds at \( p \), if the best probability of making an infinite sequence of \( a \)'s is 1”.

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1. $G_1 \overset{\text{def}}{=} \mathbb{P}_1(\nu X.\langle a \rangle X)$
   “Holds at $p$, if the best probability of making an infinite sequence of $a$’s is 1”.

2. $G_2 \overset{\text{def}}{=} \mu X.(G_1 \lor \langle b \rangle X)$
   “Best probability of reaching, by a finite sequence of $b$’s, a state where $G_1$ holds”.

**Figure**: Example of PLTS
1. \( G_1 \overset{\text{def}}{=} \mathbb{P}_{=1}(\nu X. \langle a \rangle X) \)
   "\textbf{Holds at} \( p \), if the best probability of making an infinite sequence of \( a \)'s is 1".

2. \( G_2 \overset{\text{def}}{=} \mu X. (G_1 \lor \langle b \rangle X) \)
   "Best probability of reaching, by a finite sequence of \( b \)'s, a state where \( G_1 \) holds".

3. \( G_5 \overset{\text{def}}{=} \mathbb{P}_{>0}(\mu X. (G_1 \lor \langle b \rangle X)) \)
   "Holds iff the probability (above) is greater than 0".

\[ \text{Figure: Example of PLTS} \]
\begin{align*}
1. \quad & H_1 = \nu X \cdot \mathbb{P}_{>0} \langle a \rangle X \\
& \text{“Holds if it is possible to make infinitely many possible } a \text{'s:} \\
& p \xrightarrow{a} d_1 \sim p_1 \xrightarrow{a} d_2 \sim p_2 \ldots \text{ with } d_n(p_n) > 0
\end{align*}

\begin{align*}
2. \quad & H_2 = \mu X \cdot \mathbb{P}_{=1} [a] X \\
& \text{Dual of } H_1: \text{“holds if it is impossible to make infinitely many} \\
& \text{possible } a \text{'s:}
\end{align*}

\begin{align*}
3. \quad & H_3 = \mu X \cdot ((\mathbb{P}_{>0} \langle a \rangle X) \lor \mathbb{P}_{=1} H) \\
& \text{“Holds if it is possible to make finitely many possible } a \text{'s and} \\
& \text{reach a state where } H \text{ holds with probability 1.}
\end{align*}