Commutation with Termination; Up-to Techniques for Bisimulation

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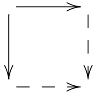
ICALP'05
July 13, 2005
Lisbon, Portugal

Outline

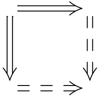
Rewriting: Confluence

- Strong confluence:



For all P, P', Q s.t. $P \rightarrow P'$ and $P \rightarrow Q$,
there exists Q' s.t. $Q \rightarrow Q'$ and $P' \rightarrow Q'$.

- Confluence:



For all P, P', Q s.t. $P \rightarrow^{\star} P'$ and $P \rightarrow^{\star} Q$,
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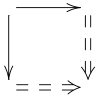
- Local confluence:



For all P, P', Q s.t. $P \rightarrow P'$ and $P \rightarrow Q$,
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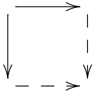

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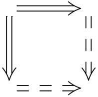

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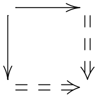

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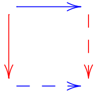
- Newman's Lemma:
  "Local confluence and termination entail confluence".

Rewriting: Commutation

- Strong commutation:



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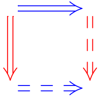

For all P, P', Q s.t. $P \rightarrow^\star P'$ and $P \rightarrow^\star Q$, there exists Q' s.t. $Q \rightarrow^\star Q'$ and $P' \rightarrow^\star Q'$.

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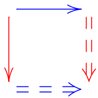

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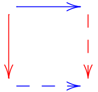

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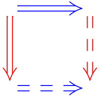

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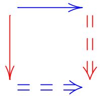

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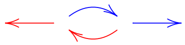
- Generalisation of Newman's Lemma ?

Rewriting: Generalisation of Newman's Lemma

Commutation is obtained from local commutation and:

- the termination of both relations: No!



Rewriting: Generalisation of Newman's Lemma

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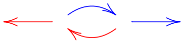

- the termination of $\rightarrow \cup \rightarrow$ (standard),

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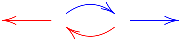

- the termination of $\rightarrow \cup \rightarrow$ (standard),
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Rewriting: Generalisation of Newman's Lemma

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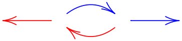
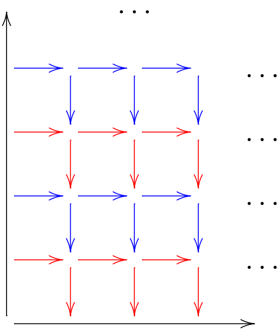


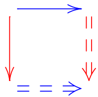
- the termination of $\rightarrow \cup \rightarrow$ (standard),
- the termination of $\rightarrow^{+}\rightarrow^{+}$ (this work).
  Neither $\rightarrow$ nor $\rightarrow$ need to terminate !

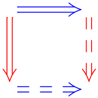


Proof sketch (1)

- If $\left\{ \begin{array}{l} \forall P, P', Q, P \textcolor{red}{\longrightarrow} P' \wedge P \textcolor{blue}{\longrightarrow} Q \\ \qquad \Rightarrow \exists Q', Q \textcolor{red}{\longrightarrow}^{\star} Q' \wedge P' \textcolor{blue}{\longrightarrow}^{\star} Q' \\ \textcolor{blue}{\longrightarrow}^{+} \textcolor{red}{\longrightarrow}^{+} \text{ terminates} \end{array} \right.$,

  then $\forall P, P', Q, P \textcolor{red}{\longrightarrow}^{\star} P' \wedge P \textcolor{blue}{\longrightarrow}^{\star} Q$
  $\Rightarrow \exists Q', Q \textcolor{red}{\longrightarrow}^{\star} Q' \wedge P' \textcolor{blue}{\longrightarrow}^{\star} Q'$.





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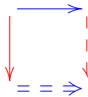
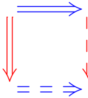
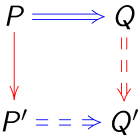
  then $\forall P, P', Q, P \longrightarrow^{\star} P' \wedge P \longrightarrow^{\star} Q$
  $\qquad \Rightarrow \exists Q', Q \longrightarrow^{\star} Q' \wedge P' \longrightarrow^{\star} Q'.$

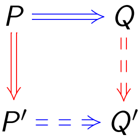




- Semi-local commutation:



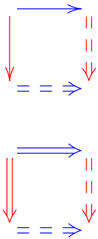
entails



Proof sketch (1)

- If $\begin{cases} \forall P, P', Q, P \textcolor{red}{\longrightarrow} P' \wedge P \textcolor{blue}{\longrightarrow} Q \\ \qquad \Rightarrow \exists Q', Q \textcolor{red}{\longrightarrow}^\star Q' \wedge P' \textcolor{blue}{\longrightarrow}^\star Q' \\ \textcolor{blue}{\longrightarrow}^+ \textcolor{red}{\longrightarrow}^+ \text{ terminates} \end{cases}$,

  then $\forall P, P', Q, P \textcolor{red}{\longrightarrow}^\star P' \wedge P \textcolor{blue}{\longrightarrow}^\star Q$
  $\Rightarrow \exists Q', Q \textcolor{red}{\longrightarrow}^\star Q' \wedge P' \textcolor{blue}{\longrightarrow}^\star Q'$.



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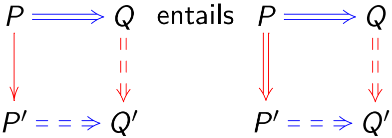
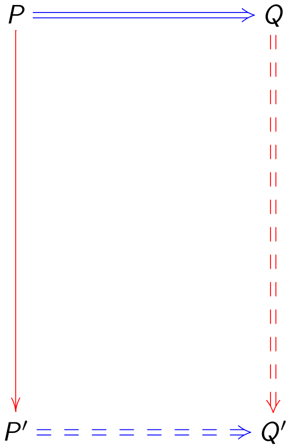


- "Stair" relation: $\textcolor{magenta}{\rightsquigarrow} \triangleq (\textcolor{blue}{\longrightarrow}^+ \textcolor{red}{\longrightarrow}^+)^+$

Proof sketch (2)

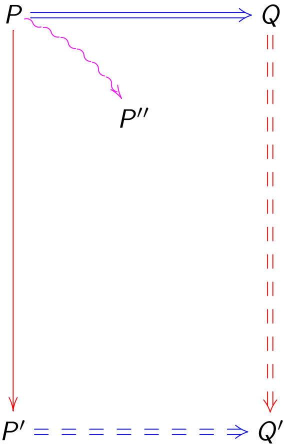
- $\theta(P):\ \forall P', Q, P \longrightarrow P' \wedge P \longrightarrow^{\star} Q$
  $\Rightarrow \exists Q', Q \longrightarrow^{\star} Q' \wedge P' \longrightarrow^{\star} Q'.$



Proof sketch (2)

- $\theta(P)$: $\forall P', Q, P \textcolor{red}{\longrightarrow} P' \wedge P \textcolor{blue}{\longrightarrow}^\star Q$
  $\Rightarrow \exists Q', Q \textcolor{red}{\longrightarrow}^\star Q' \wedge P' \textcolor{blue}{\longrightarrow}^\star Q'$.
- Well-founded induction on the termination of $\textcolor{magenta}{\rightsquigarrow} = (\textcolor{blue}{\longrightarrow}^+ \textcolor{red}{\longrightarrow}^+)^+$: we have to prove $\forall P, \phi(P) \Rightarrow \theta(P)$.

  Induction hypothesis:
  $\phi(P)$: $\forall P'', P \textcolor{magenta}{\rightsquigarrow} P'' \Rightarrow \theta(P'')$.



Proof sketch (2)

- $\theta(P)$: $\forall P', Q, P \rightarrow P' \wedge P \rightarrow^{\star} Q$
  $\Rightarrow \exists Q', Q \rightarrow^{\star} Q' \wedge P' \rightarrow^{\star} Q'$.
- Well-founded induction on the termination of $\rightsquigarrow = (\rightarrow^{+}\rightarrow^{+})^{+}$: we have to prove $\forall P, \phi(P) \Rightarrow \theta(P)$.

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  $\phi(P) : \forall P'', P \rightsquigarrow P'' \Rightarrow \theta(P'')$.
- We reformulate our goal as:
  $\forall P, Q, P \rightarrow^{\star} Q \Rightarrow \phi(P) \Rightarrow \forall P', P \rightarrow P'$
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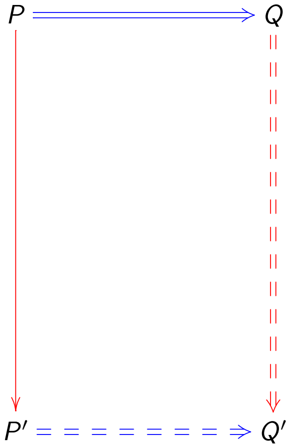

Proof sketch (2)

- $\theta(P) :\ \forall P', Q, P {\color{red}\longrightarrow} P' \wedge P {\color{blue}\longrightarrow}^\star Q$
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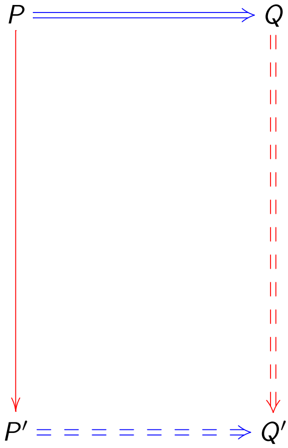
- Structural induction on $P {\color{blue}\longrightarrow}^\star Q$.



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- Well-founded induction on the termination of $\textcolor{magenta}{\rightsquigarrow} = (\textcolor{blue}{\rightarrow}^+ \textcolor{red}{\rightarrow}^+)^+$: we have to prove $\forall P, \phi(P) \Rightarrow \theta(P)$.

  External induction hypothesis:
  $\phi(P)$: $\forall P'', P \textcolor{magenta}{\rightsquigarrow} P'' \Rightarrow \theta(P'')$.
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- Structural induction on $P \textcolor{blue}{\rightarrow}^\star Q$.
- Non-trivial case: $P \textcolor{blue}{\rightarrow} P_0 \textcolor{blue}{\rightarrow}^\star Q$.

  Internal induction hypothesis:
  ψ : $\phi(P_0) \Rightarrow \forall P_1, P_0 \textcolor{red}{\rightarrow} P_1 \Rightarrow$
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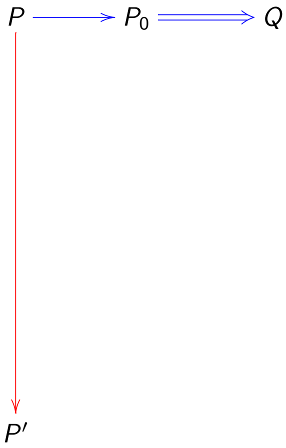

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  $\psi : \phi(P_0) \Rightarrow \forall P_1, P_0 \rightarrow P_1 \Rightarrow$
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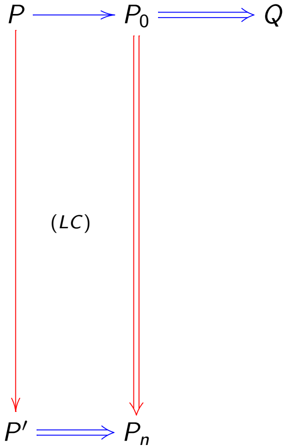

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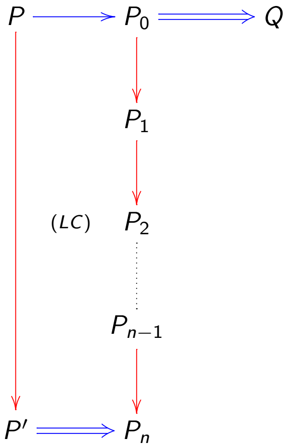

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- $\theta(P): \ \forall P', Q, P {\color{red}\longrightarrow} P' \wedge P {\color{blue}\longrightarrow}^\star Q$
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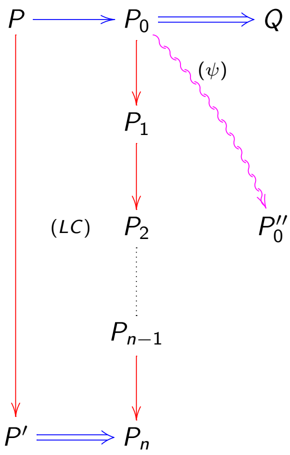

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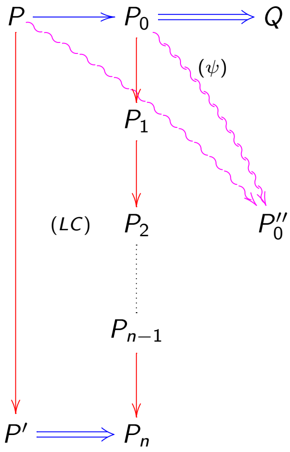

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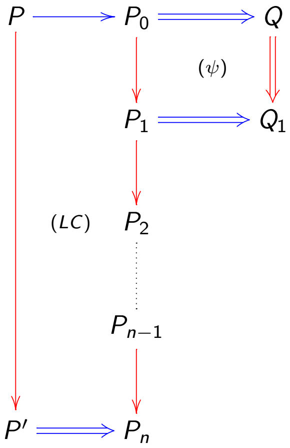

Proof sketch (2)

- $\theta(P) :\ \forall P', Q, P{\color{red}\longrightarrow}P' \wedge P{\color{blue}\longrightarrow}^{\star}Q$
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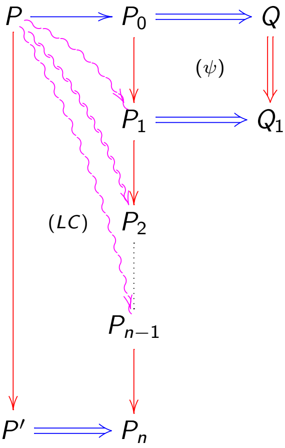

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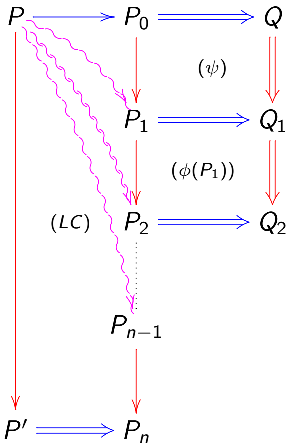

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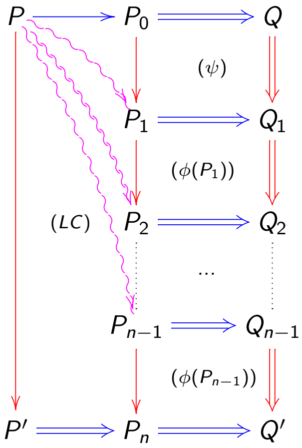

Bisimulation: Definition

- LTS: processes (P, Q..), labelled transitions ($P \xrightarrow{\alpha} P'$)
- $\mathcal{R}$ is a bisimulation if:

$$\forall P, Q, P', P \mathcal{R} Q \wedge P \xrightarrow{\alpha} P' \Rightarrow \exists Q', P' \mathcal{R} Q' \wedge Q \xrightarrow{\alpha} Q'$$

$$\begin{array}{ccc} P & \mathcal{R} & Q \\ \alpha \downarrow & & \vdots\, \alpha \\ P' & \mathcal{R} & Q' \end{array}$$

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Putting it all together

- Extend the commutation result:

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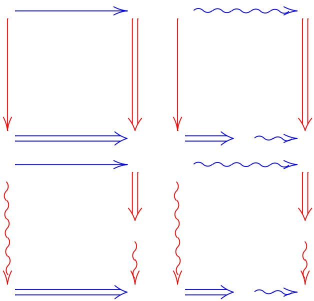
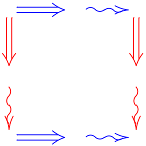


$\longrightarrow^+ \longrightarrow^+$ terminates

then



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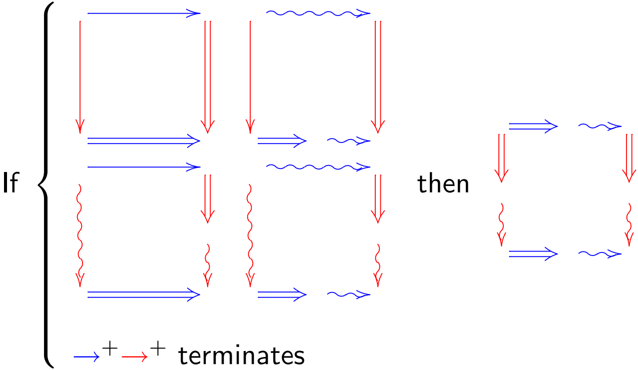

- Take $\longrightarrow = \xrightarrow{\tau}$, $\rightsquigarrow = \xrightarrow{a}\stackrel{\tau}{\Rightarrow}$, $\longrightarrow = \mathcal{B}$ and $\rightsquigarrow = \mathcal{R}$

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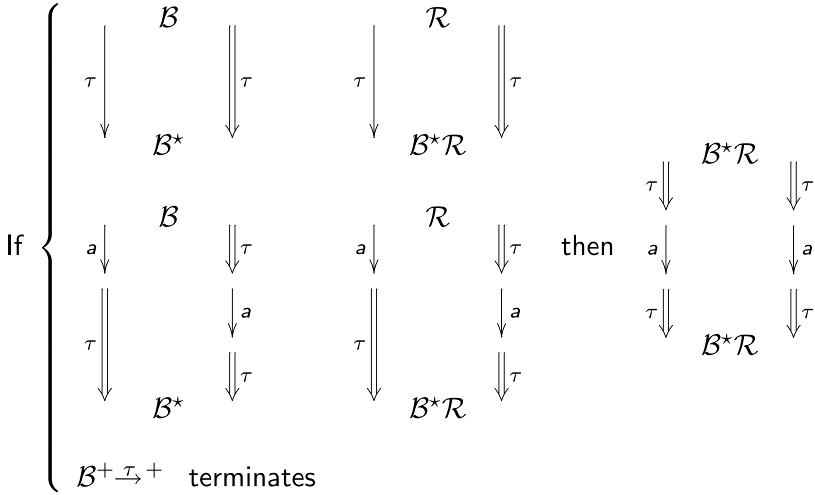

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- Developing the theory:
  - relationships with the decreasing diagrams of [van Oostrom] (already used by [Fournet] for bisimulation).
  - general extension of these techniques to the bisimulation setting?

For Further Reading I

- M. Bezem, J.W. Klop, V. van Oostrom.
  Diagram Techniques for Confluence
  Information and Computation, 141(2):172-204, 1998.
- D. Pous.
  Web appendix of this paper (with Coq proof scripts):
  http://perso.ens-lyon.fr/damien.pous/upto/.