Weak Bisimulation up to Elaboration

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Outline

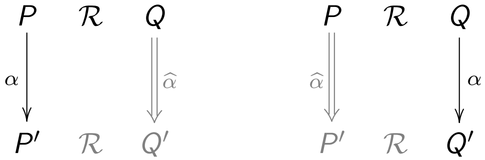
- Weak bisimilarity, Expansion, Elaboration;
- Weak bisimulation up to Elaboration;
- Up to transitivity.

Weak Bisimilarity

- LTS: processes $(P, Q..)$, labelled transitions $(P \xrightarrow{\alpha} P')$.
- Weak transitions:

$$\stackrel{\widehat{\tau}}{\Rightarrow} \triangleq \xrightarrow{\tau}^{\star} \qquad \stackrel{\widehat{a}}{\Rightarrow} \triangleq \xrightarrow{\tau}^{\star} \xrightarrow{a} \xrightarrow{\tau}^{\star} .$$

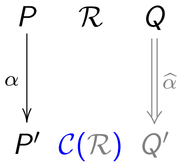
- Weak bisimulation:

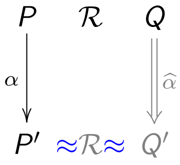


- Weak bisimilarity: $\approx \triangleq \bigcup\{\mathcal{R},\ \mathcal{R}$ weak bisimulation $\}$.

Up-to Techniques

- "Reduce the size of the bisimulation candidate."
- Up to context, up to $\approx$





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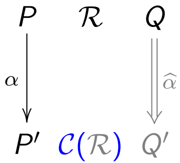
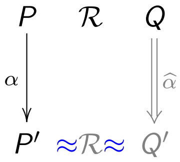
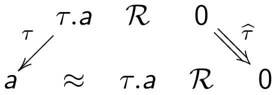




- The latter technique is not correct!

$$\mathcal{R} \triangleq \{\langle \tau.a, 0\rangle\}$$

$$a \approx \tau.a$$



Expansion [Arun-Kumar, Hennessy - Milner, Sangiorgi - 92]

- Behavioural preorder, $\succsim \subseteq \approx$ $\qquad \xrightarrow{\widehat{\tau}} \triangleq \xrightarrow{\tau}^{=} \qquad \xrightarrow{\widehat{a}} \triangleq \xrightarrow{a}$

$$\begin{array}{ccc} P & \succsim & Q \\ \alpha\downarrow & & \downarrow\widehat{\alpha} \\ P' & \succsim & Q' \end{array} \qquad\qquad \begin{array}{ccc} P & \succsim & Q \\ \widehat{\alpha}\Downarrow & & \downarrow\alpha \\ P' & \succsim & Q' \end{array}$$

- $P \succsim Q$: Q must move "faster" than P, at each step.

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- $P \succsim Q$: Q must move "faster" than P, at each step.
- Weak bisimulation up to expansion:

$$\begin{array}{ccc} P & \mathcal{R} & Q \\ \alpha\big\downarrow & & \big\Downarrow\widehat{\alpha} \\ P' & \succsim\mathcal{R}\precsim & Q' \end{array} \qquad\qquad (\mathcal{R} \text{ symmetric})$$

- This technique has been used in complex proofs (π, Join. . .).

Elaboration [Arun-Kumar, Natarajan - 95]

- Behavioural preorder, $\succapprox \subseteq \approx$ $\qquad \stackrel{\tau}{\Rightarrow} \triangleq \stackrel{\tau}{\rightarrow}^{+} \qquad \stackrel{a}{\Rightarrow} \triangleq \stackrel{\widehat{a}}{\Rightarrow}$

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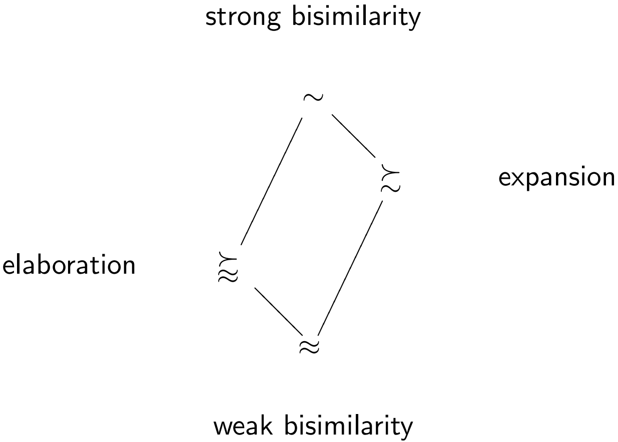
- Elaboration in CCS:
  - Close to $\approx$: if $P \approx Q$, then $!\tau|P \succapprox Q$.
  - Contains *progressing bisimulation*: the largest bisimulation which is a congruence [Montanari, Sassone - 91].
  - Almost a precongruence (preemptive power of τ).

A Picture


strong bisimilarity
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weak bisimilarity

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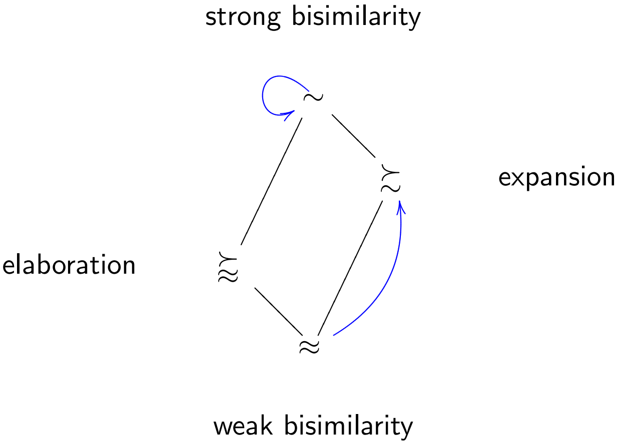
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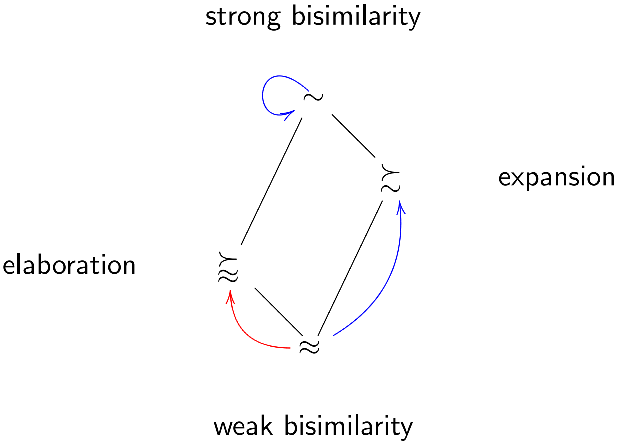
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Weak Bisimulation up to Elaboration?

If $\mathcal{R}$ is symmetric and
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then $\gtrapprox \mathcal{R} \lessapprox$ is a bisimulation, and hence $\mathcal{R} \subseteq \approx$

Proof.

$$\begin{array}{ccccccc} P & \gtrapprox & P_0 & \mathcal{R} & Q_0 & \lessapprox & Q \\ \tau\downarrow & & & & & & \\ P' & & & & & & \end{array}$$

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- $P > P_1$ for some well-founded partial order $>$.

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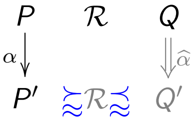

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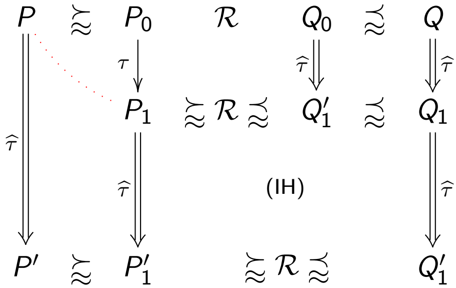

- $P > P_1$ for some well-founded partial order $>$.
- If $\xrightarrow{\tau}$ terminates, then so does $\gtrapprox \xrightarrow{\tau}$.

Termination Hypothesis

- If $\xrightarrow{\tau}$ terminates, then so does $\succapprox \xrightarrow{\tau}$.

$$
\begin{array}{ccccccc}
P_0 & \succapprox & Q_0 & & & & \\
 & & \downarrow \tau & & & & \\
 & & P_1 & \succapprox & Q_1 & & \\
 & & & & \downarrow \tau & & \\
 & & & & P_2 & \succapprox & Q_2 \\
 & & & & & & \ldots
\end{array}
$$

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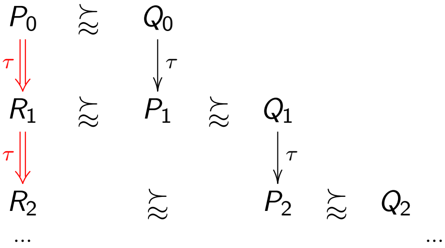

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- The same reasoning does not hold with $\approx$.

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\end{array}
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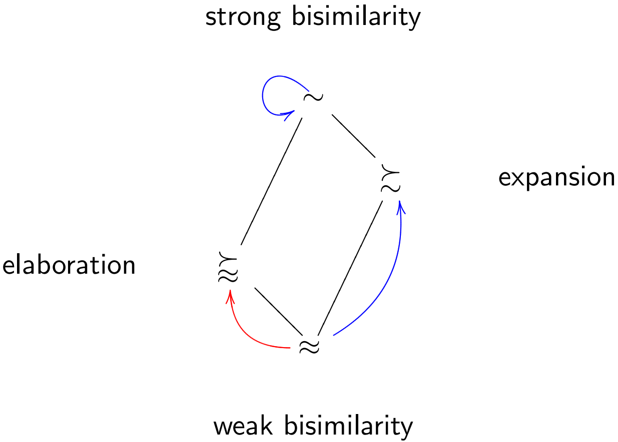
- The same reasoning does not hold with $\approx$.
- **Theorem.** *If $\xrightarrow{\tau}$ terminates, then weak bisimulation up to elaboration is correct.*

A Picture



(when $\xrightarrow{\tau}$ terminates)

Up to Transitivity (for strong bisimilarity)

If $\mathcal{R}$ is symmetric and
$$\begin{array}{ccc} P & \mathcal{R} & Q \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ P' & \mathcal{R}^\star & Q' \end{array}$$
then $\mathcal{R}^\star$ is a strong bisimulation.

- Subsumes the "strong up to strong" technique:

  If
  $$\begin{array}{ccc} P & \mathcal{S} & Q \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ P' & \sim \mathcal{S} \sim & Q' \end{array}$$
  then $\mathcal{R} \triangleq \mathcal{S} \cup \sim$ satisfies the diagram above, and $\mathcal{S} \subseteq \mathcal{R} \subseteq \sim$.

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- Not correct with weak bisimilarity games!
- Only correct on one side for expansion:

  If $\begin{array}{ccc} P & \mathcal{R} & Q \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\widehat{\alpha}} \\ P' & \mathcal{R}^\star & Q' \end{array}$ and $\begin{array}{ccc} P & \mathcal{R} & Q \\ {\scriptstyle\widehat{\alpha}}\Downarrow & & \downarrow{\scriptstyle\alpha} \\ P' & \mathcal{R} & Q' \end{array}$ then $\mathcal{R} \subseteq \gtrsim$.

Up to Transitivity for Elaboration

- **Theorem.** *If $\xrightarrow{\tau}$ terminates, then up to transitivity is correct for elaboration, i.e.:*

$$\text{If}\quad \begin{array}{ccc} P & \mathcal{R} & Q \\ {\scriptstyle\alpha}\big\downarrow & & \big\Downarrow{\scriptstyle\widehat{\alpha}} \\ P' & \mathcal{R}^{\star} & Q' \end{array} \quad\text{and}\quad \begin{array}{ccc} P & \mathcal{R} & Q \\ {\scriptstyle\alpha}\big\Downarrow & & \big\downarrow{\scriptstyle\alpha} \\ P' & \mathcal{R}^{\star} & Q' \end{array} \quad\text{then } \mathcal{R} \subseteq \succapprox\ .$$

Proof sketch. Prove that $\mathcal{R}^{\star} \overset{\tau}{\Rightarrow}$ terminates, then reason by well-founded induction.

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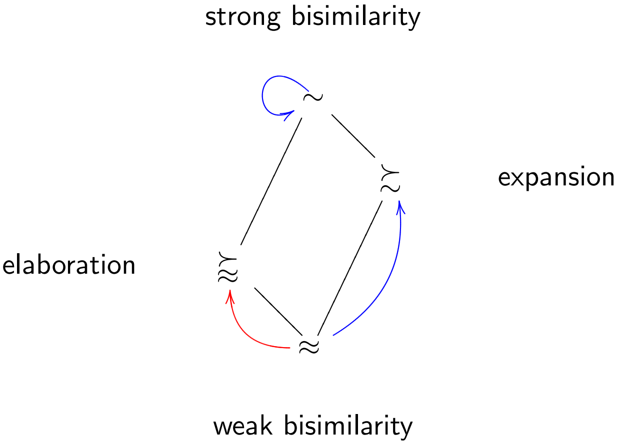
- **Corollary.** (a variant of Newman's Lemma for $\approx$):

$$\textit{If } \xrightarrow{\tau} \textit{ terminates and } \begin{array}{ccc} P & \xrightarrow{\quad\tau\quad} & Q \\ {\scriptstyle\alpha}\big\downarrow & & \big\Downarrow{\scriptstyle\widehat{\alpha}} \\ P' & \overset{\widehat{\tau}}{\Longrightarrow} & Q' \end{array} \quad \textit{then } \xrightarrow{\tau} \subseteq \approx \ .$$

The Final Picture



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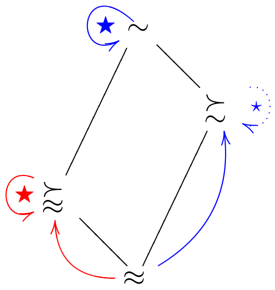

(when $\xrightarrow{\tau}$ terminates)

Concluding Remarks

- [ICALP'05]: "if $\mathcal{B}^\star$ is a bisimulation and $\mathcal{B}^+ \xrightarrow{\tau}^+$ terminates, then $\mathcal{B}^\star$ is a correct up to technique for $\approx$".
  - The termination hypothesis mentions the relation $\mathcal{B}$.
  - Lacks compositionality: we cannot define the "largest $\mathcal{B}$".

- The up to elaboration technique appears as a much more tractable instantiation of the above results:
  - coinductively defined,
  - up to transitivity, up to context.

- Is it reasonable to ask for the termination of $\xrightarrow{\tau}$?
  - For most algorithms, termination is part of the correctness.
  - When loops are needed (busy waiting), isolate terminating silent actions.

- We still need to demonstrate the use of these techniques on some non-trivial examples.