

Normal Form Bisimulations for Delimited-Control Operators

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Abstract. We define a notion of normal form bisimilarity for the untyped call-by-value λ -calculus extended with the delimited-control operators shift and reset. Normal form bisimilarities are simple, easy-to-use behavioral equivalences, which relate terms without having to test them within all contexts (like contextual equivalence), or by applying them to function arguments (like applicative bisimilarity). We prove that normal form bisimilarity is sound but not complete w.r.t. contextual equivalence, and we define up-to techniques, which aim at simplifying bisimulation proofs. Finally, we illustrate the simplicity of the techniques we develop by proving several equivalences on terms.

1 Introduction

Morris-style contextual equivalence [15] is usually considered as the most natural behavioral equivalence for functional languages based on λ -calculi. Roughly, two terms are equivalent if we can exchange one for the other in a bigger program without affecting its behavior (i.e., whether it terminates or not). The quantification over program contexts makes contextual equivalence hard to use in practice and, therefore, it is common to look for easier-to-use behavioral equivalence, such as *bisimilarities*.

Several kinds of bisimilarity relations have been defined so far, such as *applicative bisimilarity* [1], *normal form bisimilarity* [11], and *environmental bisimilarity* [16]. Applicative and environmental bisimilarities usually compare terms by applying them to function arguments; as a result, we obtain relations which completely characterize contextual equivalence, but still contain a universal quantification over arguments in their definitions. In contrast, normal form bisimilarity does not need such quantification; it equates terms by reducing them to normal form, and by requiring the sub-terms of these normal forms to be bisimilar. Normal form relations are convenient in practice, but they are usually not *complete* w.r.t. contextual equivalence, i.e., there exist contextually equivalent terms that are not normal form bisimilar.

A notion of normal form bisimulation has been defined in various calculi, including the pure λ -calculus [10, 11], the λ -calculus with ambiguous choice [12], the $\lambda\mu$ -calculus [13], and the $\lambda\mu\rho$ -calculus [18], where normal form bisimilarity characterizes contextual equivalence. However, it has not yet been defined for calculi with *delimited-control* operators, such as shift and reset [6]—programming

constructs rapidly gaining currency in the recent years. Unlike abortive control operators (such as *call/cc*), delimited-control operators allow to delimit access to the current continuation and to compose continuations. The operators *shift* and *reset* were introduced as a direct-style realization of the traditional success/failure continuation model of backtracking otherwise expressible only in continuation-passing style [6]. The numerous theoretical and practical applications of *shift* and *reset* (see, e.g., [2] for an extensive list) include the seminal result by Filinski showing that a programming language endowed with *shift* and *reset* is monadically complete [7].

Up to now, only an applicative bisimilarity have been defined for a calculus with *shift* and *reset* [4]. In this paper, we define several notions of normal form bisimilarity, more tractable than contextual equivalence or applicative bisimilarity. We prove they are *sound* w.r.t. contextual equivalence (i.e., included in contextual equivalence), but fail to be complete. We also develop *up-to techniques*, which help when proving equivalences with normal form bisimulations.

In Section 2, we define the λ -calculus with delimited control we use in this paper, and we recall the definition of contextual equivalence of [4]. We then define in Section 3 the main notion of normal form bisimilarity, and prove its properties. In Section 4, we refine the definition of normal form bisimilarity to relate more contextually equivalent terms, at the cost of extra complexity in bisimulation proofs. We also propose several up-to techniques which simplify the proofs of equivalence of terms. In Section 5, we illustrate the simplicity of use (compared to applicative bisimilarity) of the notions we define by employing them in the proofs of several equivalences of terms. Section 6 concludes the paper, and Appendix A contains the congruence proofs of the considered normal form bisimilarities.

2 The Calculus $\lambda_{\mathcal{S}}$

In this section, we present the syntax, reduction semantics, and contextual equivalence of the language $\lambda_{\mathcal{S}}$ studied throughout this article.

2.1 Syntax

The language $\lambda_{\mathcal{S}}$ extends the call-by-value λ -calculus with the delimited-control operators *shift* and *reset* [6]. We assume we have a set of term variables, ranged over by x, y, z , and k . We use the metavariable k for term variables representing a continuation (e.g., when bound with a *shift*), while x, y , and z stand for any values; we believe such distinction helps to understand examples and reduction rules. The syntax of terms and values is given by the following grammars:

$$\begin{aligned} \text{Terms: } t &::= x \mid \lambda x.t \mid tt \mid \mathcal{S}k.t \mid \langle t \rangle \\ \text{Values: } v &::= \lambda x.t \mid x \end{aligned}$$

The operator *shift* ($\mathcal{S}k.t$) is a capture operator, the extent of which is determined by the delimiter *reset* ($\langle \cdot \rangle$). A λ -abstraction $\lambda x.t$ binds x in t and a *shift* construct

$Sk.t$ binds k in t ; terms are equated up to α -conversion of their bound variables. The set of free variables of t is written $\text{fv}(t)$; a term is *closed* if it does not contain free variables.

We distinguish several kinds of contexts, as follows.

$$\begin{array}{l} \text{Pure contexts:} \quad E ::= \square \mid v E \mid E t \\ \text{Evaluation contexts:} \quad F ::= \square \mid v F \mid F t \mid \langle F \rangle \\ \text{Contexts:} \quad C ::= \square \mid \lambda x.C \mid t C \mid C t \mid Sk.C \mid \langle C \rangle \end{array}$$

Regular contexts are ranged over by C . The pure evaluation contexts¹ (abbreviated as pure contexts), ranged over by E , represent delimited continuations and can be captured by the shift operator. The call-by-value evaluation contexts, ranged over by F , represent arbitrary continuations and encode the chosen reduction strategy. Contexts are interpreted inside-out [2], i.e., \square stands for the empty context, $v E$ represents the term with a hole $E[v \]]$, $E t$ represents $E[\] t]$, $\langle F \rangle$ represents $F[\langle \]]$, etc. Filling a context C (respectively E , F) with a term t produces a term, written $C[t]$ (respectively $E[t]$, $F[t]$); the free variables of t may be captured in the process. A context is *closed* if it contains only closed terms.

2.2 Reduction Semantics

Before we present the reduction semantics for λ_S , let us briefly describe an intuitive semantics of shift and reset by means of an example written in SML using Filinski's implementation of shift and reset [7].

Example 1. The following function copies a list [3], where the SML expression `shift (fn k => t)` corresponds to $Sk.t$ and `reset (fn () => t)` corresponds to $\langle t \rangle$:

```
fun copy xs =
  let fun visit nil = nil
      | visit (x::xs) = visit (shift (fn k => x :: (k xs)))
  in reset (fn () => visit xs) end
```

This simple function illustrates the main ideas of programming with shift and reset:

- The control delimiter `reset` delimits continuations. Any control effects occurring in the subsequent calls to function `visit` are local to function `copy`.
- The control operator `shift` captures delimited continuations. Each, but last, recursive call to `visit` abstracts the continuation that can be represented as a function `fn v => reset (fn () => visit v)` and binds it to `k`.
- Captured continuations are composed statically. When applied, in the expression `x :: (k xs)`, the captured continuation becomes the current delimited continuation that is isolated from the rest of the program, and in particular from the expression `x ::`, by a control delimiter—witness the control

¹ This terminology comes from Kameyama (e.g., in [8]).

delimiter in the expression `fn v => reset (fn () => visit v)` representing the captured continuation.

Formally, the call-by-value reduction semantics of λ_S is defined as follows, where $t\{v/x\}$ is the usual capture-avoiding substitution of v for x in t :

$$\begin{array}{l} (\beta_v) \quad F[(\lambda x.t) v] \rightarrow_v F[t\{v/x\}] \\ (\text{shift}) \quad F[\langle E[\mathcal{S}k.t] \rangle] \rightarrow_v F[\langle t\{\lambda x.\langle E[x] \rangle/k\} \rangle] \text{ with } x \notin \text{fv}(E) \\ (\text{reset}) \quad F[\langle v \rangle] \rightarrow_v F[v] \end{array}$$

The term $(\lambda x.t) v$ is the usual call-by-value redex for β -reduction (rule (β_v)). The operator $\mathcal{S}k.t$ captures its surrounding context E up to the dynamically nearest enclosing `reset`, and substitutes $\lambda x.\langle E[x] \rangle$ for k in t (rule (shift)). If a `reset` is enclosing a value, then it has no purpose as a delimiter for a potential capture, and it can be safely removed (rule (reset)). All these reductions may occur within a metalevel context F . The chosen call-by-value evaluation strategy is encoded in the grammar of the evaluation contexts.

Example 2. Let $i = \lambda x.x$ and $\omega = \lambda x.x x$. We present the sequence of reductions initiated by $\langle\langle(\mathcal{S}k_1.i(k_1 i)) \mathcal{S}k_2.\omega\rangle\rangle(\omega \omega)$. The term $\mathcal{S}k_1.i(k_1 i)$ is within the pure context $E = (\square(\omega \omega)) \mathcal{S}k_2.\omega$ (remember that we represent contexts inside-out), enclosed in a delimiter $\langle \cdot \rangle$, so E is captured according to rule (shift) .

$$\langle\langle(\mathcal{S}k_1.i(k_1 i)) \mathcal{S}k_2.\omega\rangle\rangle(\omega \omega) \rightarrow_v \langle i((\lambda x.\langle(x \mathcal{S}k_2.\omega)(\omega \omega))\rangle) i) \rangle$$

The role of `reset` in $\lambda x.\langle E[x] \rangle$ is more clear after reduction of the (β_v) -redex $(\lambda x.\langle E[x] \rangle) i$.

$$\langle i((\lambda x.\langle(x \mathcal{S}k_2.\omega)(\omega \omega))\rangle) i) \rangle \rightarrow_v \langle i \langle\langle i \mathcal{S}k_2.\omega \rangle\rangle(\omega \omega) \rangle$$

When the captured context E is reactivated, it is not *merged* with the context $i \square$, but *composed* thanks to the `reset` enclosing E . As a result, the capture triggered by $\mathcal{S}k_2.\omega$ leaves the term i outside the first enclosing `reset` untouched.

$$\langle i \langle\langle i \mathcal{S}k_2.\omega \rangle\rangle(\omega \omega) \rangle \rightarrow_v \langle i \langle \omega \rangle \rangle$$

Because k_2 does not occur in ω , the context $i(\square(\omega \omega))$ is discarded when captured by $\mathcal{S}k_2.\omega$. Finally, we remove the useless delimiter $\langle i \langle \omega \rangle \rangle \rightarrow_v \langle i \omega \rangle$ with rule (reset) , and we then (β_v) -reduce and remove the last delimiter $\langle i \omega \rangle \rightarrow_v \langle \omega \rangle \rightarrow_v \omega$. Note that, while the reduction strategy is call-by-value, some function arguments are not evaluated, like the non-terminating term $\omega \omega$ in this example.

There exist terms which are not values and which cannot be reduced any further; these are called *stuck terms*.

Definition 1. A term t is stuck if t is not a value and $t \not\rightarrow_v$.

For example, the term $E[\mathcal{S}k.t]$ is stuck because there is no enclosing `reset`; the capture of E by the shift operator cannot be triggered. In fact, stuck terms are easy to characterize.

Lemma 1. *A term t is stuck iff $t = E[Sk.t']$ for some E , k , and t' or $t = F[xv]$ for some F , x , and v .*

We call *control stuck terms* the terms of the form $E[Sk.t]$ and *open stuck terms* the terms of the form $F[xv]$.

Definition 2. *A term t is a normal form, if t is a value or a stuck term.*

We call *redexes* (ranged over by r) the terms of the form $(\lambda x.t) v$, $\langle E[Sk.t] \rangle$, and $\langle v \rangle$. Thanks to the following unique-decomposition property, the reduction \rightarrow_v is deterministic.

Lemma 2. *For all terms t , either t is a normal form, or there exist a unique redex r and a unique context F such that $t = F[r]$.*

Finally, we write \rightarrow_v^* for the transitive and reflexive closure of \rightarrow_v , and we define the evaluation relation of λ_S as follows.

Definition 3. *We write $t \Downarrow_v t'$ if $t \rightarrow_v^* t'$ and $t' \nrightarrow_v$.*

The result of the evaluation of a term, if it exists, is a normal form. If a term t admits an infinite reduction sequence, we say it *diverges*, written $t \uparrow_v$. In the rest of the article, we use extensively $\Omega = (\lambda x.x x) (\lambda x.x x)$ as an example of such a term.

2.3 Contextual Equivalence

In this paper, we use the same contextual equivalence as in [4], where control stuck terms can be observed.

Definition 4. *Let t_0, t_1 be terms. We write $t_0 \approx_c t_1$ if for all C such that $C[t_0]$ and $C[t_1]$ are closed, the following hold:*

- $C[t_0] \Downarrow_v v_0$ implies $C[t_1] \Downarrow_v v_1$;
- $C[t_0] \Downarrow_v t'_0$, where t'_0 is control stuck, implies $C[t_1] \Downarrow_v t'_1$, with t'_1 control stuck as well;

and conversely for $C[t_1]$.

We have proved in [4] the following context lemma [14] for λ_S , which helps proving that terms are contextually equivalent. Instead of testing terms with (free-variables capturing) general contexts, we can simply first close them and then put them within (closed) evaluation contexts.

Lemma 3 (Context Lemma). *We have $t_0 \approx_c t_1$ iff for all closed contexts F and for all substitutions σ such that $t_0\sigma$ and $t_1\sigma$ are closed, the following hold:*

- $F[t_0\sigma] \Downarrow_v v_0$ implies $F[t_1\sigma] \Downarrow_v v_1$;
- $F[t_0\sigma] \Downarrow_v t'_0$, where t'_0 is control stuck, implies $F[t_1\sigma] \Downarrow_v t'_1$, with t'_1 control stuck as well;

and conversely for $F[t_1\sigma]$.

In the rest of the paper, when proving that terms are contextually equivalent, we implicitly use Lemma 3.

3 Normal Form Bisimilarity

In this section, we discuss a notion of bisimulation based on the evaluation of terms to normal forms. The difficulties are mainly in the handling of control stuck terms, and in the definition of the relation on non-pure evaluation contexts. We propose here a first way to deal with control stuck terms, that will be refined in the next section. In any definitions or proofs, we say a variable is *fresh* if it does not appear free in the terms or contexts under consideration.

3.1 Definition

Following Lassen’s approach [11], we define a normal form bisimulation where we relate terms by comparing the result of their respective evaluations (if it exists). As we need to compare evaluation contexts as well as terms, we extend a relation \mathcal{R} on terms to contexts in the following way: we write $F_0 \mathcal{R} F_1$ if $F_0 = F_0'[\langle E_0[] \rangle]$, $F_1 = F_1'[\langle E_1[] \rangle]$, $F_0'[x] \mathcal{R} F_1'[x]$, and $\langle E_0[x] \rangle \mathcal{R} \langle E_1[x] \rangle$ for a fresh x , or $F_0 = E_0$, $F_1 = E_1$, and $E_0[x] \mathcal{R} E_1[x]$ for a fresh x . The rationale behind this definition is explained later.

Definition 5. *A relation \mathcal{R} on terms is a normal form bisimulation if $t_0 \mathcal{R} t_1$ implies:*

- if $t_0 \Downarrow_v v_0$, then there exists v_1 such that $t_1 \Downarrow_v v_1$ and $v_0 x \mathcal{R} v_1 x$ for a fresh variable x ;
- if $t_0 \Downarrow_v E_0[\mathcal{S}k.t'_0]$, then there exist t'_1 and E_1 such that $t_1 \Downarrow_v E_1[\mathcal{S}k.t'_1]$, $\langle t'_0 \rangle \mathcal{R} \langle t'_1 \rangle$, and $E_0 \mathcal{R} E_1$;
- if $t_0 \Downarrow_v F_0[xv_0]$, then there exist F_1 and v_1 such that $t_1 \Downarrow_v F_1[xv_1]$, $v_0 \mathcal{R} v_1$, and $F_0 \mathcal{R} F_1$;
- the converse of the above conditions on t_1 .

Normal form bisimilarity, written \approx , is the largest normal form bisimulation.

Henceforth, we often drop the “normal form” attribute when talking about bisimulations for brevity. Two terms t_0 and t_1 are bisimilar if their evaluations lead to matching normal forms (e.g., if t_0 evaluates to a control stuck term, then so does t_1) with bisimilar sub-components. We now detail the different cases.

Normal form bisimilarity does not distinguish between evaluation to a variable and evaluation to a λ -abstraction. Instead, we relate terms evaluating to any values v_0 and v_1 by comparing $v_0 x$ and $v_1 x$, where x is fresh. As originally pointed out by Lassen [11], this is necessary for the bisimilarity to be sound w.r.t. η -expansion; otherwise it would distinguish η -equivalent terms such as $\lambda y.x y$ and x .

For a control stuck term $E_0[\mathcal{S}k.t'_0]$ to be executed, it has to be plugged into an evaluation context surrounded by a reset; by doing so, we obtain a term of the form $\langle t'_0 \{ \lambda x. \langle E_0'[x] \rangle / k \} \rangle$ for some context E_0' . Notice that the resulting term is within a reset; similarly, when comparing $E_0[\mathcal{S}k.t'_0]$ and $E_1[\mathcal{S}k.t'_1]$, we ask for the shift bodies t'_0 and t'_1 to be related when surrounded by a reset. We also

compare E_0 and E_1 , which amounts to executing $E_0[x]$ and $E_1[x]$ for a fresh x , since the two contexts are pure. Comparing t'_0 and t'_1 without reset would be too discriminating, as it would distinguish the two contextually equivalent terms $Sk.\langle t \rangle$ and $Sk.t$.² Indeed, without reset, we would have to relate $\langle t \rangle$ and t , which are not equivalent in general (take $t = Sk'.v$ for some v), while Definition 5 requires $\langle\langle t \rangle\rangle$ and $\langle t \rangle$ to be related (which holds for all t ; see Example 3).

Two normal forms $F_0[xv_0]$ and $F_1[xv_1]$ are bisimilar if the values v_0 and v_1 as well as the contexts F_0 and F_1 are related. We have to be careful when defining bisimilarity on (possibly non pure) evaluation contexts. We cannot simply relate F_0 and F_1 by executing $F_0[y]$ and $F_1[y]$ for a fresh y . Such a definition would equate the contexts \square and $\langle \square \rangle$, which in turn would relate the terms xv and $\langle xv \rangle$, which are not contextually equivalent: they are distinguished by the context $(\lambda x.\square)\lambda y.Sk.\Omega$. A context containing a reset enclosing the hole should be related only to contexts with the same property. However, we do not want to precisely count the number of delimiters around the hole; doing so would distinguish $\langle \square \rangle$ and $\langle\langle \square \rangle\rangle$, and therefore it would discriminate the contextually equivalent terms $\langle xv \rangle$ and $\langle\langle xv \rangle\rangle$. Hence, the extension of bisimulation to contexts (given before Definition 5) checks that if one of the contexts contains a reset surrounding the hole, then so does the other; then it compares the contexts beyond the first enclosing delimiter by simply evaluating them using a fresh variable. As a result, it rightfully distinguishes \square and $\langle \square \rangle$, but it relates $\langle \square \rangle$ and $\langle\langle \square \rangle\rangle$.

Example 3. We prove that $\langle t \rangle \approx \langle\langle t \rangle\rangle$ by showing that $\mathcal{R} = \{\langle\langle t \rangle\rangle, \langle t \rangle\} \cup \approx$ is a bisimulation. If $\langle t \rangle \Downarrow_v v$, then $\langle\langle t \rangle\rangle \Downarrow_v v$, and $v x \approx v x$ holds for a fresh x . The case $\langle t \rangle \Downarrow_v E[Sk.t']$ is not possible; one can check that if $\langle t \rangle \rightarrow_v t'$, then t' is a value, or can be written $\langle t'' \rangle$ for some t'' (and similarly for $\langle t \rangle \Downarrow_v t'$).

If $\langle t \rangle \Downarrow_v F[xv]$, then there exists F' such that $t \Downarrow_v F'[xv]$ and $F = \langle F'[\] \rangle$. Therefore, we have $\langle\langle t \rangle\rangle \Downarrow_v \langle\langle F'[xv] \rangle\rangle$. We have $v \approx v$, and we have to prove that $\langle F'[\] \rangle \mathcal{R} \langle\langle F'[\] \rangle\rangle$ to conclude. If F' is a pure context E , then we have to prove $\langle E[y] \rangle \mathcal{R} \langle\langle E[y] \rangle\rangle$ and $y \mathcal{R} \langle y \rangle$ for a fresh y , which are both true because $\approx \subseteq \mathcal{R}$. If $F' = F''[\langle E[\] \rangle]$, then given a fresh y , we have to prove $\langle F''[y] \rangle \mathcal{R} \langle\langle F''[y] \rangle\rangle$ (clear by the definition of \mathcal{R}), and $\langle E[y] \rangle \mathcal{R} \langle E[y] \rangle$ (true because $\approx \subseteq \mathcal{R}$).

Similarly, it is easy to check that the evaluations of $\langle\langle t \rangle\rangle$ are matched by $\langle t \rangle$.

Example 4. In [5], the authors propose variants of the call-by-value Curry's and Turing's fixed point combinators using shift and reset. Let $\theta = \lambda xy.y(\lambda z.xxyz)$. We prove that Turing's combinator $t_0 = \theta\theta$ is bisimilar to its shift and reset variant $t_1 = \langle \theta Sk.kk \rangle$. We build the candidate relation \mathcal{R} incrementally, starting from (t_0, t_1) . Evaluating t_0 and t_1 , we obtain $t_0 \Downarrow_v \lambda y.y(\lambda z.\theta\theta yz) = v_0$ and $t_1 \Downarrow_v \lambda y.y(\lambda z.(\lambda x.\langle \theta x \rangle)(\lambda x.\langle \theta x \rangle) yz) = v_1$. To relate the bodies of v_0 and v_1 , we must add $(v'_0 = \lambda z.\theta\theta yz, v'_1 = \lambda z.(\lambda x.\langle \theta x \rangle)(\lambda x.\langle \theta x \rangle) yz)$ to \mathcal{R} (as well as (x, x) for a fresh x). Evaluating the bodies of v'_0 and v'_1 , we obtain respectively yv'_0z and yv'_1z ; to relate these two normal forms, we just need to add (xz, xz)

² The equivalence $Sk.\langle t \rangle \equiv Sk.t$ comes from Kameyama and Hasegawa's axiomatization of shift and reset [8] and has been proved using applicative bisimilarity in [4]

to \mathcal{R} , since we already have $v'_0 \mathcal{R} v'_1$. One can check that the constructed relation \mathcal{R} is a normal form bisimulation.

In contrast, Curry's combinator $t'_0 = \lambda x. \delta_x \delta_x$, where $\delta_x = \lambda y. x (\lambda z. y y z)$, is not bisimilar to its delimited-control variant $t'_1 = \lambda x. \langle \delta_x \mathcal{S}k.k k \rangle$. Indeed, evaluating the bodies of the two values, we obtain respectively $x (\lambda z. \delta_x \delta_x z)$ and $\langle \langle x (\lambda z. (\lambda y. \langle \delta_x y \rangle) (\lambda y. \langle \delta_x y \rangle) z) \rangle \rangle$, and these open stuck terms are not bisimilar, because $\square \not\approx \langle \langle \square \rangle \rangle$. In fact, t'_0 and t'_1 are distinguished by the context $\square \lambda x. \mathcal{S}k.\Omega$. Finally, we can prove that the two original combinators $\theta \theta$ and $\lambda x. \delta_x \delta_x$ are bisimilar, using the same bisimulation as in [11].

3.2 Soundness and Completeness

Usual congruence proofs for normal form bisimilarities include direct proofs, where a context and/or substitutive closure of the bisimilarity is proved to be itself a bisimulation [10, 12, 18], and proofs based on continuation-passing style (CPS) translations [11, 13]. The CPS approach consists in proving a CPS-based correspondence between the bisimilarity \mathcal{R}_1 we want to prove sound and a relation \mathcal{R}_2 that we already know it is a congruence. Because CPS translations are usually themselves compatible, we can then conclude that \mathcal{R}_1 is a congruence. For example, in the λ -calculus, Lassen proved a CPS-correspondence between the eager normal form bisimilarity and the Böhm trees equivalence [11].

Because shift and reset have been originally defined in terms of CPS [6], one can expect the CPS approach to be successful. However, the CPS translation of shift and reset assumes that λ_S terms are executed within an outermost reset, and therefore they cannot evaluate to a control stuck term. For the normal form bisimilarity to be sound w.r.t. CPS, we would have to restrict its definition to terms of the form $\langle t \rangle$. This does not seem possible while keeping Definition 5 without quantification on contexts. For example, to relate values v_0 and v_1 , we would have to execute $v_0 x$ and $v_1 x$ (where x is fresh) under reset. However, requiring simply $\langle v_0 x \rangle$ and $\langle v_1 x \rangle$ to be related would be unsound; such a definition would relate $\lambda y. \mathcal{S}k.k y$ and $\lambda y. \mathcal{S}k.(\lambda z.z) y$, which can be distinguished by the context $\langle \square \Omega \rangle$. To be sound, we would have to require $\langle E[v_0 x] \rangle$ to be related to $\langle E[v_1 x] \rangle$ for every E ; we then introduce a quantification on contexts we want to avoid in the first place. Because normal forms may contain control stuck terms as sub-terms, normal form bisimilarity has to be able to handle them, and, therefore, it cannot be restricted to terms of the form $\langle t \rangle$ only.

Since CPS cannot help us in proving congruence, we follow a more direct approach, by relying on a context closure. Given a relation \mathcal{R} , we define its substitutive and context closure by the rules of Fig. 1. The main lemma of the congruence proof is then as follows.

Lemma 4. *If \mathcal{R} is a normal form bisimulation, then so is $\widehat{\mathcal{R}}$.*

More precisely, we prove that if $t_0 \widehat{\mathcal{R}} t_1$ and t_0 evaluates to some normal form t'_0 in m steps, then t_1 evaluates to a normal form t'_1 such that $t'_0 \widehat{\mathcal{R}} t'_1$. The proof is by nested induction on m and on the definition of $\widehat{\mathcal{R}}$; it can be found in Appendix A. Congruence of \approx then follows immediately.

$$\begin{array}{c}
\frac{t_0 \mathcal{R} t_1}{t_0 \widehat{\mathcal{R}} t_1} \quad \frac{t_0 \widehat{\mathcal{R}} t_1 \quad v_0 \widehat{\mathcal{R}} v_1}{t_0\{v_0/x\} \widehat{\mathcal{R}} t_1\{v_1/x\}} \quad \frac{t_0 \widehat{\mathcal{R}} t_1 \quad E_0[x] \widehat{\mathcal{R}} E_1[x] \quad x \text{ fresh}}{E_0[t_0] \widehat{\mathcal{R}} E_1[t_1]} \\
\frac{t_0 \widehat{\mathcal{R}} t_1 \quad \langle E_0[x] \rangle \widehat{\mathcal{R}} \langle E_1[x] \rangle \quad F_0[x] \widehat{\mathcal{R}} F_1[x] \quad x \text{ fresh}}{F_0[\langle E_0[t_0] \rangle] \widehat{\mathcal{R}} F_1[\langle E_1[t_1] \rangle]} \quad \frac{t_0 \widehat{\mathcal{R}} t_1}{\lambda x.t_0 \widehat{\mathcal{R}} \lambda x.t_1} \\
\frac{t_0 \widehat{\mathcal{R}} t_1}{Sk.t_0 \widehat{\mathcal{R}} Sk.t_1}
\end{array}$$

Fig. 1: Context and substitutive closure of a relation \mathcal{R}

Corollary 1. *The relation \approx is a congruence*

We can then easily prove that \approx is sound w.r.t. contextual equivalence.

Theorem 1. *We have $\approx \subseteq \approx_c$.*

The following counter-example shows that the inclusion is in fact strict; normal form bisimilarity is not complete.

Proposition 1. *Let $i = \lambda y.y$. We have $\langle\langle x i \rangle Sk.i \rangle \approx_c \langle\langle x i \rangle (\langle x i \rangle Sk.i) \rangle$, but $\langle\langle x i \rangle Sk.i \rangle \not\approx \langle\langle x i \rangle (\langle x i \rangle Sk.i) \rangle$.*

Proof. Replacing x by a closed value v , we get $\langle\langle v i \rangle Sk.i \rangle$ and $\langle\langle v i \rangle (\langle v i \rangle Sk.i) \rangle$, which both evaluate to i if the evaluation of $\langle v i \rangle$ terminates (otherwise, they both diverge). With this observation, it is easy to prove that $\langle\langle x i \rangle Sk.i \rangle$ and $\langle\langle x i \rangle (\langle x i \rangle Sk.i) \rangle$ are contextually equivalent. They are not bisimilar, because the terms $\langle y Sk.i \rangle$ and $\langle y (\langle x i \rangle Sk.i) \rangle$ (where y is fresh) are not bisimilar: the former evaluates to i while the latter is in normal form (but is not a value). \square

4 Refined Bisimilarity and Up-to Techniques

In this section, we propose an improvement of the definition of normal form bisimilarity, and we discuss some proof techniques which aim at simplifying equivalence proofs.

4.1 Refined Bisimilarity

Normal form bisimilarity could better deal with control stuck terms. To illustrate this, consider the following terms.

Proposition 2. *Let $i = \lambda x.x$. We have $Sk.i \approx_c (Sk.i) \Omega$, but $Sk.i \not\approx (Sk.i) \Omega$.*

Proof. If $Sk.i$ and $(Sk.i) \Omega$ are put within a pure context, then we obtain two control stuck terms, and if we put these two terms within a context $F[\langle E[\] \rangle]$, then they both reduce to $F[i]$. Therefore, $Sk.i$ and $(Sk.i) \Omega$ are contextually equivalent. They are not normal form bisimilar, since the contexts \square and $\square \Omega$ are not bisimilar (x converges while $x \Omega$ diverges). \square

When comparing control stuck terms, normal form bisimilarity considers contexts and shift bodies separately, while they are combined if the control stuck terms are put under a reset and the capture goes through. To fix this issue, we consider an other definition of bisimulation, given below.

Definition 6. A relation \mathcal{R} on terms is a refined normal form bisimulation if $t_0 \mathcal{R} t_1$ implies:

- if $t_0 \Downarrow_v E_0[\mathcal{S}k.t'_0]$, then there exist t'_1 and E_1 such that $t_1 \Downarrow_v E_1[\mathcal{S}k.t'_1]$, and $\langle t'_0 \{ \lambda x. \langle k' E_0[x] \rangle / k \} \rangle \mathcal{R} \langle t'_1 \{ \lambda x. \langle k' E_1[x] \rangle / k \} \rangle$ for fresh k' and x ;
- the same clauses as in Definition 5 for the other normal forms;
- the converse of the above conditions on t_1 .

Refined bisimilarity, written $\overset{\bullet}{\approx}$, is the largest refined bisimulation.

In the control stuck terms case, Definition 6 simulates the capture of E_0 (respectively E_1) by $\mathcal{S}k.t_0$ (respectively $\mathcal{S}k.t_1$). However, if t_0 is put into a context $\langle E[] \rangle$, then $\mathcal{S}k.t'_0$ captures a context bigger than E_0 , namely $E[E_0[]]$. We take such possibility into account by using a variable k' in the bisimulation clause, which represents the context that can be captured beyond E_0 and E_1 .

Refined bisimilarity contains the regular bisimilarity.

Proposition 3. We have $\approx \subset \overset{\bullet}{\approx}$.

Indeed, for control stuck terms, we have $t_0 \Downarrow_v E_0[\mathcal{S}k.t'_0]$, $t_1 \Downarrow_v E_1[\mathcal{S}k.t'_1]$, $E_0 \approx E_1$, and $\langle t'_0 \rangle \approx \langle t'_1 \rangle$. Because \approx is a congruence (Corollary 1), it is easy to see that $\langle t'_0 \{ \lambda x. \langle k' E_0[x] \rangle / k \} \rangle \approx \langle t'_1 \{ \lambda x. \langle k' E_1[x] \rangle / k \} \rangle$ holds a fresh k' . Therefore, $\overset{\bullet}{\approx}$ is a refined bisimulation, and is included in $\overset{\bullet}{\approx}$. The inclusion is strict, because $\overset{\bullet}{\approx}$ relates the terms of Proposition 2, while \approx does not.

Proving that $\overset{\bullet}{\approx}$ is sound requires some adjustments to the congruence proof of \approx . First, given a relation \mathcal{R} on terms, we define its substitutive, bisimilar, and context closure $\widetilde{\mathcal{R}}$ by extending the rules of Fig. 1 with the following one.

$$\frac{t_0 \overset{\bullet}{\approx} \widetilde{\mathcal{R}} \overset{\bullet}{\approx} t_1}{t_0 \widetilde{\mathcal{R}} t_1}$$

Our goal is to prove that $\overset{\bullet}{\approx}$ is a refined bisimilarity. To this end, we need a few lemmas.

Lemma 5. If $x \notin \text{fv}(E)$, then $(\lambda x. E[x]) t \approx E[t]$.

One can prove that $\{((\lambda x. E[x]) t, E[t]), x \notin \text{fv}(E)\} \cup \{(t, t)\}$ is a bisimulation, by a straightforward case analysis on the result of the evaluation of t (if it exists). Note that Lemma 5, known as the β_Ω axiom in [8], has also been proved in [4] using applicative bisimulation. We can see that the proof is much simpler using normal form bisimulation. With Lemma 5, congruence of \approx , and Proposition 3, we then prove the following result.

Lemma 6. *If $x \notin \text{fv}(E_0) \cup \text{fv}(E_1)$ and $y \notin \text{fv}(E_1)$ then $\langle t\{\lambda x.\langle E_1[E_0[x]]\rangle/k\} \overset{\bullet}{\approx} \langle t\{\lambda x.\langle (\lambda y.E_1[y]) E_0[x]\rangle/k\}.$*

The main lemma of the congruence proof of $\overset{\bullet}{\approx}$ is as follows.

Lemma 7. *If \mathcal{R} is a refined bisimulation, then so is $\widetilde{\mathcal{R}}$.*

The proof is an adaptation of the proof of Lemma 4. We sketch one sub-case of the proof, to illustrate why we need $\widetilde{\mathcal{R}}$ (instead of $\widehat{\mathcal{R}}$) and Lemma 6.

Proof (Sketch). Assume we are in the case where $E_0[t_0] \widetilde{\mathcal{R}} E_1[t_1]$ with $E_0[y] \widetilde{\mathcal{R}} E_1[y]$ for a fresh y , and $t_0 \widetilde{\mathcal{R}} t_1$. Moreover, suppose $t_0 \Downarrow_v E_0'[\mathcal{S}k.t_0]$. Then by induction hypothesis, we know that there exist E_1', t_1' such that $t_1 \Downarrow_v E_1'[\mathcal{S}k.t_1]$, and $\langle t_0'\{\lambda x.\langle k' E_0'[x]\rangle/k\} \widetilde{\mathcal{R}} \langle t_1'\{\lambda x.\langle k' E_1'[x]\rangle/k\} \text{ (*)}$ for a fresh k' . Hence, we have $E_0[t_0] \Downarrow_v E_0[E_0'[\mathcal{S}k.t_0]]$ and $t_1 \Downarrow_v E_1[E_1'[\mathcal{S}k.t_1]]$, and we want to prove that $\langle t_0'\{\lambda x.\langle k' E_0[E_0'[x]]\rangle/k\} \widetilde{\mathcal{R}} \langle t_1'\{\lambda x.\langle k' E_1[E_1'[x]]\rangle/k\} \text{ holds. Because } E_0[y] \widetilde{\mathcal{R}} E_1[y]$, we have $\lambda y.k' E_0[y] \widetilde{\mathcal{R}} \lambda y.k' E_1[y] \text{ (**)}$. Using (*) and (**), we obtain $\langle t_0'\{\lambda x.\langle (\lambda y.k' E_0[y]) E_0'[x]\rangle/k\} \widetilde{\mathcal{R}} \langle t_1'\{\lambda x.\langle (\lambda y.k' E_1[y]) E_1'[x]\rangle/k\}$, because $\widetilde{\mathcal{R}}$ is substitutive. By Lemma 6, we know that

$$\begin{aligned} \langle t_0'\{\lambda x.\langle k' E_0[E_0'[x]]\rangle/k\} \overset{\bullet}{\approx} \langle t_0'\{\lambda x.\langle (\lambda y.k' E_0[y]) E_0'[x]\rangle/k\} \\ \langle t_1'\{\lambda x.\langle k' E_1[E_1'[x]]\rangle/k\} \overset{\bullet}{\approx} \langle t_1'\{\lambda x.\langle (\lambda y.k' E_1[y]) E_1'[x]\rangle/k\}, \end{aligned}$$

which means that $\langle t_0'\{\lambda x.\langle k' E_0[E_0'[x]]\rangle/k\} \overset{\bullet}{\approx} \widetilde{\mathcal{R}} \overset{\bullet}{\approx} \langle t_1'\{\lambda x.\langle k' E_1[E_1'[x]]\rangle/k\} \text{ holds. The required result then holds because } \overset{\bullet}{\approx} \widetilde{\mathcal{R}} \overset{\bullet}{\approx} \subseteq \widetilde{\mathcal{R}}. \quad \square$

We can then conclude that $\overset{\bullet}{\approx}$ is a congruence, and is sound w.r.t. \approx_c .

Corollary 2. *The relation $\overset{\bullet}{\approx}$ is a congruence.*

Theorem 2. *We have $\overset{\bullet}{\approx} \subseteq \approx_c$.*

The inclusion is strict, because the terms of Proposition 1 are still not related by $\overset{\bullet}{\approx}$.

We would like to stress that even if $\overset{\bullet}{\approx}$ equates more contextually equivalent terms than \approx , the latter is still useful, since it leads to very simple proofs of equivalence, as we can see with Lemma 5 (and with the examples of Section 5). Therefore, $\overset{\bullet}{\approx}$ does not disqualify \approx as a proof technique.

4.2 Up-to Techniques

The idea behind up-to techniques [17, 9, 16] is to define relations that are not exactly bisimulations but are included in bisimulations. It usually leads to definitions of simpler candidate relations and to simpler bisimulation proofs. As pointed out in [9], using a direct approach to prove congruence of the normal form bisimilarity (as in Sections 3.2 and 4.1) makes up-to techniques based on the context closure easy to define and to prove valid. For example, we define bisimulation up to substitutive and context closure (in short, up to context) as follows.

Definition 7. A relation \mathcal{R} on terms is a bisimulation up to context if $t_0 \mathcal{R} t_1$ implies:

- if $t_0 \Downarrow_v v_0$, then there exists v_1 such that $t_1 \Downarrow_v v_1$ and $v_0 x \widehat{\mathcal{R}} v_1 x$ for a fresh variable x ;
- if $t_0 \Downarrow_v E_0[\text{Sk}.t'_0]$, then there exist t'_1 and E_1 such that $t_1 \Downarrow_v E_1[\text{Sk}.t'_1]$, $\langle t'_0 \rangle \widehat{\mathcal{R}} \langle t'_1 \rangle$, and $E_0 \widehat{\mathcal{R}} E_1$;
- if $t_0 \Downarrow_v F_0[x v_0]$, then there exist F_1 and v_1 such that $t_1 \Downarrow_v F_1[x v_1]$, $v_0 \widehat{\mathcal{R}} v_1$, and $F_0 \widehat{\mathcal{R}} F_1$;
- the converse of the above conditions on t_1 .

Similarly, we can define a notion of refined bisimulation up to context by using the refined clause for control stuck terms, and by replacing $\widehat{\mathcal{R}}$ by $\widetilde{\mathcal{R}}$ in the above definition. The proofs of Lemmas 4 and 7 can easily be adapted to bisimulations up to context; a trivial change is needed only in the inductive case where $t_0 \widehat{\mathcal{R}} t_1$ (respectively $t_0 \widetilde{\mathcal{R}} t_1$) comes from $t_0 \mathcal{R} t_1$.

Lemma 8. If \mathcal{R} is a bisimulation up to context, then $\widehat{\mathcal{R}}$ is a bisimulation.

If \mathcal{R} is a refined bisimulation up to context, then $\widetilde{\mathcal{R}}$ is a refined bisimulation.

Consequently, if \mathcal{R} is a bisimulation up to context, and if $t_0 \mathcal{R} t_1$, then $t_0 \approx t_1$, because $\mathcal{R} \subseteq \widehat{\mathcal{R}} \subseteq \approx$.

Example 5. We can simplify the proof of bisimilarity between Turing's fixed point combinator and its delimited-control variant (cf. Example 4); indeed, it is enough to prove that $\mathcal{R} = \{(\theta \theta, \langle \theta \text{Sk}.k k \rangle), (\theta \theta, \lambda x. \langle \theta x \rangle \lambda x. \langle \theta x \rangle)\}$ is a bisimulation up to context.

When proving equivalence of terms, it is sometimes easier to reason in a small-step fashion instead of trying to evaluate terms completely. To allow this kind of reasoning, we define the following small-step notion.

Definition 8. A relation \mathcal{R} on terms is small-step bisimulation up to context if $t_0 \mathcal{R} t_1$ implies:

- if $t_0 \rightarrow_v t'_0$, then there exists t'_1 such that $t_1 \rightarrow_v^* t'_1$ and $t'_0 \widehat{\mathcal{R}} t'_1$;
- if $t_0 = v_0$, then there exists v_1 such that $t_1 \Downarrow_v v_1$ and $v_0 x \widehat{\mathcal{R}} v_1 x$ for a fresh variable x ;
- if $t_0 = E_0[\text{Sk}.t'_0]$, then there exist t'_1 and E_1 such that $t_1 \Downarrow_v E_1[\text{Sk}.t'_1]$, $\langle t'_0 \rangle \widehat{\mathcal{R}} \langle t'_1 \rangle$, and $E_0 \widehat{\mathcal{R}} E_1$;
- if $t_0 = F_0[x v_0]$, then there exist F_1 and v_1 such that $t_1 \Downarrow_v F_1[x v_1]$, $v_0 \widehat{\mathcal{R}} v_1$, and $F_0 \widehat{\mathcal{R}} F_1$;
- the converse of the above conditions on t_1 .

Similarly, we can define the refined variant. Again, it is easy to check the validity of these two proof techniques.

Lemma 9. If \mathcal{R} is a small-step bisimulation up to context, then $\widehat{\mathcal{R}}$ is a bisimulation.

If \mathcal{R} is a refined small-step bisimulation up to context, then $\widetilde{\mathcal{R}}$ is a refined bisimulation.

We show how these relations can be used in the next section (Proposition 5).

5 Examples

We now prove several equivalences of terms to illustrate the usefulness of the relations and techniques defined in this paper. The examples of this section are taken from [8], where they are derived from the axiomatization of λ_S . In the following, we write \mathcal{I} for the identity bisimulation $\{(t, t)\}$.

Lemma 5 already witnesses that proving that two terms are normal form bisimilar can be quite simple, especially compared to applicative bisimilarity [4]. The following example is of the same kind.

Proposition 4. *If $x \notin \text{fv}(E)$, then $E[(\lambda x.t_0) t_1] \approx (\lambda x.E[t_0]) t_1$.*

Proof. By showing that $\{(E[(\lambda x.t_0) t_1], (\lambda x.E[t_0]) t_1), x \notin \text{fv}(E)\} \cup \mathcal{I}$ is a normal form bisimulation. The proof is straightforward by case analysis on the result of the evaluation of t_0 (if it exists). \square

The next example demonstrates how useful small-step relations can be.

Proposition 5. *If $x \notin \text{fv}(E)$, then $\langle (\lambda x.\langle E[x] \rangle) t \rangle \approx \langle E[t] \rangle$.*

Proof. Let $\mathcal{R} = \{(\langle (\lambda x.\langle E[x] \rangle) t \rangle, \langle E[t] \rangle), x \notin \text{fv}(E)\}$. We prove that $\mathcal{R} \cup \approx$ is a small-step bisimulation up to context, by case analysis on t .

- If $t \rightarrow_v t'$, then $\langle (\lambda x.\langle E[x] \rangle) t \rangle \rightarrow_v \langle (\lambda x.\langle E[x] \rangle) t' \rangle$, $\langle E[t] \rangle \rightarrow_v \langle E[t'] \rangle$, and we have $\langle (\lambda x.\langle E[x] \rangle) t' \rangle \mathcal{R} \langle E[t'] \rangle$, as required.
- If $t = v$, then $\langle (\lambda x.\langle E[x] \rangle) v \rangle \rightarrow_v \langle \langle E[v] \rangle \rangle$. We have proved in Example 3 that $\langle \langle E[v] \rangle \rangle \approx \langle E[v] \rangle$.
- If $t = F[y v]$, then we have to relate $\langle (\lambda x.\langle E[x] \rangle) F[\] \rangle$ and $\langle E[F[\]] \rangle$ (we clearly have $v \approx v$). If $F = F'[\langle E'[\] \rangle]$, then we have $\langle (\lambda x.\langle E[x] \rangle) F'[z] \rangle \mathcal{R} \langle E[F'[z]] \rangle$ and $\langle E'[z] \rangle \approx \langle E'[z] \rangle$ for a fresh z . If $F = E'$, then we have $\langle (\lambda x.\langle E[x] \rangle) E'[z] \rangle \mathcal{R} \langle E[E'[z]] \rangle$ for a fresh z .
- If $t = E'[\mathcal{S}k.t']$, then $\langle (\lambda x.\langle E[x] \rangle) t \rangle \rightarrow_v \langle t' \{ \lambda y. \langle (\lambda x.\langle E[x] \rangle) E'[y] \} / k \} \rangle$, and $\langle E[t] \rangle \rightarrow_v \langle t' \{ \lambda y. \langle E[E'[y]] \rangle / k \} \rangle$. We have $\langle (\lambda x.\langle E[x] \rangle) E'[y] \rangle \mathcal{R} \langle E[E'[y]] \rangle$, therefore $\langle t' \{ \lambda y. \langle (\lambda x.\langle E[x] \rangle) E'[y] \} / k \} \rangle \widehat{\mathcal{R}} \langle t' \{ \lambda y. \langle E[E'[y]] \rangle / k \} \rangle$ holds, as wished. \square

Without using small-step bisimulation, the definition of the candidate relation \mathcal{R} as well as the bisimulation proof would be much more complex, as we would have to compute the results of the evaluations of $\langle (\lambda x.\langle E[x] \rangle) t \rangle$ and $\langle E[t] \rangle$, which is particularly difficult if t is a control stuck term.

For the next example, we have to use refined bisimilarity.

Proposition 6. *If $k' \notin \text{fv}(E) \cup \text{fv}(t)$ and $x \notin \text{fv}(E)$, then we have $E[\mathcal{S}k.t] \overset{\bullet}{\approx} \mathcal{S}k'.t \{ \lambda x. \langle k' E[x] \rangle / k \}$.*

Proof. The two terms are control stuck terms, therefore, we have to prove $\langle t \{ \lambda x. \langle k'' E[x] \rangle / k \} \rangle \overset{\bullet}{\approx} \langle t \{ \lambda x. \langle (\lambda y. \langle k'' y \rangle) E[x] \rangle / k \} \rangle$ for a fresh k'' . We know that $\langle k'' E[x] \rangle \approx \langle (\lambda y. \langle k'' y \rangle) E[x] \rangle$ holds by Proposition 5. Consequently, we have $\langle k'' E[x] \rangle \overset{\bullet}{\approx} \langle (\lambda y. \langle k'' y \rangle) E[x] \rangle$ by Proposition 3. We can then conclude by congruence of $\overset{\bullet}{\approx}$. \square

Without Proposition 5, we would have to prove $\langle k'' E[x] \rangle \approx \langle (\lambda y. \langle k'' y \rangle) E[x] \rangle$ directly, using a small-step refined bisimulation up to context. Proving Proposition 6 with the regular normal form bisimilarity would require us to equate $E[y]$ and y (where y is fresh), which is not possible if $E = (\lambda z. \Omega)$ \square .

6 Conclusion

In this paper, we propose several normal formal bisimilarities and up-to techniques for a λ -calculus with shift and reset, and we demonstrate their usefulness on several examples. Proving equivalences of terms with the regular normal form bisimilarity generates minimal proof obligations, especially when used in conjunction with (small-step) up-to context techniques. If the regular bisimilarity fails to relate the tested terms, then the refined bisimilarity can be of help. If they both fail, then we may have to use the applicative bisimilarity [4], which, unlike the bisimilarities of this paper, is complete.

We believe this work can easily be adapted to other delimited-control operators. Another possible future work would be to define *environmental bisimulations* [16] for λ_S . When comparing two terms, environmental relations use an additional component, the environment, which represents the current knowledge of the observer. For example, in the pure λ -calculus, when two tested terms reduce to values, they become known to the observer and are added to the environment. The observer can then challenge two λ -abstractions by applying them to two related arguments built from the environment. Environmental bisimilarities are usually sound and complete, and also allow for up-to techniques.

Another issue is to find a characterization of contextual equivalence for λ -calculi with abortive control operators. Normal form bisimilarities have been defined for extensions of the $\lambda\mu$ -calculus [13], but they are usually not complete, except in the presence of a store construct [18]. It might be possible to reach completeness with applicative or environmental bisimilarities.

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A Soundness Proof

Lemma 10. *If $t \rightarrow_v t'$ then $t\{v/x\} \rightarrow_v t'\{v/x\}$.*

Proof. We proceed by case analysis on $t \rightarrow_v t'$.

Suppose $F[(\lambda y.t_0)]v_0 \rightarrow_v F[t_0\{v_0/y\}]$. We have

$$\begin{aligned} t\{v/x\} &= F\{v/x\}[(\lambda y.t_0\{v/x\})v_0\{v/x\}] \\ &\rightarrow_v F\{v/x\}[t_0\{v/x\}\{v_0\{v/x\}/y\}] = t'\{v/x\}, \end{aligned}$$

as required.

Suppose $F[\langle E_0[Sk.t_0] \rangle] \rightarrow_v F[\langle t_0\{\lambda y.\langle E_0[y] \rangle/k \rangle \rangle]$. We have

$$\begin{aligned} t\{v/x\} &= F\{v/x\}[\langle E_0\{v/x\}[Sk.t_0\{v/x\}] \rangle] \\ &\rightarrow_v F\{v/x\}[\langle t_0\{v/x\}\{\lambda y.\langle E_0\{v/x\}[y] \rangle/k \rangle \rangle] = t'\{v/x\}, \end{aligned}$$

as required.

Suppose $F[\langle v_0 \rangle] \rightarrow_v F[v_0]$. We have

$$t\{v/x\} = F\{v/x\}[\langle v_0\{v/x\} \rangle] \rightarrow_v F\{v/x\}[v_0\{v/x\}],$$

as required. □

Lemma 11. *Let \mathcal{R} be a bisimulation. We have $v_0 \widehat{\mathcal{R}} v_1$ iff $v_0 x \widehat{\mathcal{R}} v_1 x$ for a fresh x , with derivations of the same size (and similarly for $\widetilde{\mathcal{R}}$).*

If $\lambda x.t_0 \widehat{\mathcal{R}} \lambda x.t_1$ then $t_0 \widehat{\mathcal{R}} t_1$ (and similarly for $\widetilde{\mathcal{R}}$).

Proof. Easy by induction on the derivation of $\widehat{\mathcal{R}}$. □

Lemma 12 (Lemma 4 in the paper). *If \mathcal{R} is a bisimulation, then $\widehat{\mathcal{R}}$ is a bisimulation.*

Proof. Because $\widehat{\mathcal{R}}$ is symmetric, we only have to prove that it is a simulation. We consider $t_0 \widehat{\mathcal{R}} t_1$ with t_0 evaluating in m steps; we prove that t_1 evaluates to a term related by $\widehat{\mathcal{R}}$ by induction on m , and on the size of the derivation of $t_0 \widehat{\mathcal{R}} t_1$, ordered lexicographically. The base case is easy; we treat only the general case. Note that the cases $\lambda x.t_0 \widehat{\mathcal{R}} \lambda x.t_1$ with $t_0 \widehat{\mathcal{R}} t_1$ and $Sk.t_0 \widehat{\mathcal{R}} Sk.t_1$ with $t_0 \widehat{\mathcal{R}} t_1$ are not treated there since they are part of the base case.

Assume we have $t_0 \mathcal{R} t_1$. This case is easy because \mathcal{R} is a bisimulation and $\mathcal{R} \subseteq \widehat{\mathcal{R}}$.

Assume $t_0\{v_0/x\} \widehat{\mathcal{R}} t_1\{v_1/x\}$ with $t_0 \widehat{\mathcal{R}} t_1$ and $v_0 \widehat{\mathcal{R}} v_1$. We suppose first that $t_0\{v_0/x\} \Downarrow_v v'_0$. We have two cases to consider.

- If $t_0 \Downarrow_v v''_0$, then $v'_0 = v''_0\{v_0/x\}$ by Lemma 10. By induction hypothesis, there exists v''_1 such that $t_1 \Downarrow_v v''_1$, and $v''_0 y \widehat{\mathcal{R}} v''_1 y$ holds for a fresh y . By Lemma 10, we have $t_1\{v_1/x\} \Downarrow_v v''_1\{v_1/x\}$, and we also have $v''_0\{v_0/x\}y \widehat{\mathcal{R}} v''_1\{v_1/x\}y$ for a fresh y , hence the result holds.

- Suppose $t_0 \Downarrow_v F_0[x v_0'']$ with $F_0\{v_0/x\}[v_0 v_0''\{v_0/x\}] \Downarrow_v v_0'$. By induction hypothesis, there exist F_1, v_1'' such that $t_1 \Downarrow_v F_1[x v_1'']$, $F_0 \widehat{\mathcal{R}} F_1$, and $v_0'' \widehat{\mathcal{R}} v_1''$. Because $F_0\{v_0/x\}[v_0 v_0''\{v_0/x\}]$ evaluates to a value, v_0 must be a λ -abstraction $v_0 = \lambda y.t_0'$. Assume v_1 is a variable y . By Lemma 11, we have $v_0 z \widehat{\mathcal{R}} y z$ for a fresh z , so by induction hypothesis there exist E_0', v_0''' such that $v_0 z \Downarrow_v E_0'[y v_0''']$, $E_0'[x'] \widehat{\mathcal{R}} x'$ for a fresh x' , and $v_0''' \widehat{\mathcal{R}} z$. Consequently, we have

$$F_0\{v_0/x\}[v_0 v_0''\{v_0/x\}] \Downarrow_v F_0\{v_0/x\}[E_0'\{v_0''\{v_0/x\}/z\}[y v_0'''\{v_0''\{v_0/x\}/z\}]],$$

which is in contradiction with $F_0\{v_0/x\}[v_0 v_0''\{v_0/x\}] \Downarrow_v v_0'$. Therefore, v_1 must be a λ -abstraction $\lambda y.t_1'$.

By Lemma 11, we have $t_0' \widehat{\mathcal{R}} t_1'$. The reductions

$$F_0\{v_0/x\}[v_0 v_0''\{v_0/x\}] \rightarrow_v F_0\{v_0/x\}[t_0'\{v_0''\{v_0/x\}/y\}]$$

and

$$F_1\{v_1/x\}[v_1 v_1''\{v_1/x\}] \rightarrow_v F_1\{v_1/x\}[t_1'\{v_1''\{v_1/x\}/y\}]$$

hold, and because $F_0\{v_0/x\}[t_0'\{v_0''\{v_0/x\}/y\}] \widehat{\mathcal{R}} F_1\{v_1/x\}[t_1'\{v_1''\{v_1/x\}/y\}]$, and $F_0\{v_0/x\}[t_0'\{v_0''\{v_0/x\}/y\}]$ evaluates to v_0' in $m - 1$ steps or less, by induction hypothesis, there exists v_1' such that $F_1\{v_1/x\}[t_1'\{v_1''\{v_1/x\}/y\}] \Downarrow_v v_1'$ and $v_0' z \widehat{\mathcal{R}} v_1' z$ for a fresh z . Because $t_1\{v_1/x\} \Downarrow_v v_1'$, we have the required result.

The case $t_0\{v_0/x\} \Downarrow_v E_0[Sk.t_0']$ is treated similarly. Suppose $t_0\{v_0/x\} \Downarrow_v F_0[y v_0']$ with $y \neq x$. We have two possible cases. The case $t_0 \Downarrow_v F_0'[y v_0']$ is similar to the case $t_0\{v_0/x\} \Downarrow_v v_0'$ with $t_0 \Downarrow_v v_0$. Suppose $t_0 \Downarrow_v F_0'[x v_0']$ with $F_0'\{v_0/x\}[v_0 v_0''\{v_0/x\}] \Downarrow_v F_0[y v_0']$. By induction hypothesis, there exist F_1', v_1'' such that $t_1 \Downarrow_v F_1'[x v_1'']$, $F_0' \widehat{\mathcal{R}} F_1'$, and $v_0'' \widehat{\mathcal{R}} v_1''$. If both v_0 and v_1 are λ -abstractions, then we proceed as in the case $t_0\{v_0/x\} \Downarrow_v v_0'$ with $t_0 \Downarrow_v F_0[x v_0']$. If both v_0 and v_1 are variables, then we must have $v_0 = v_1 = y$, and the required result holds. Suppose v_0 is a variable and v_1 is a λ -abstraction (the symmetric case is treated similarly). Then we must have $v_0 = y$, $F_0 = F_0'\{v_0/x\}$, and $v_0' = v_0''\{v_0/x\}$. By Lemma 11, we have $v_0 z \widehat{\mathcal{R}} v_1 z$ for a fresh z , so by induction hypothesis there exist E_1', v_1''' such that $y z \Downarrow_v E_1'[y v_1''']$, $E_1'[x'] \widehat{\mathcal{R}} x'$ for a fresh x' , and $v_1''' \widehat{\mathcal{R}} z$. Consequently, we have $t_1\{v_1/x\} \Downarrow_v F_1'\{v_0/x\}[E_1'\{v_1'''\{v_1/x\}/z\}[y v_1'''\{v_1'''\{v_1/x\}/z\}]]$. From the relations $F_0' \widehat{\mathcal{R}} F_1'$ and $x' \widehat{\mathcal{R}} E_1'[x']$, we deduce $F_0'\{v_0/x\} \widehat{\mathcal{R}} F_1'\{v_0/x\}[E_1'\{v_1'''\{v_1/x\}/z\}[]]$, and from $v_0'' \widehat{\mathcal{R}} v_1''$ and $z \widehat{\mathcal{R}} v_1'''$, we deduce $v_0''\{v_0/x\} \widehat{\mathcal{R}} v_1'''\{v_1'''\{v_1/x\}/z\}$. Consequently we have the required result.

Assume $E_0[t_0] \widehat{\mathcal{R}} E_1[t_1]$ with $t_0 \widehat{\mathcal{R}} t_1$ and $E_0[x] \widehat{\mathcal{R}} E_0[x]$ for a fresh x . Suppose $E_0[t_0] \Downarrow_v v_0$. Then $t_0 \Downarrow_v v_0'$ and $E_0[v_0'] \Downarrow_v v_0$. By induction hypothesis, there exists v_1' such that $t_1 \Downarrow_v v_1'$, and $v_0' y \widehat{\mathcal{R}} v_1' y$ for a fresh y . We have $v_0' \widehat{\mathcal{R}} v_1'$ by Lemma 11. Because $E_0[v_0'] \Downarrow_v$, there exists a normal form t_0' such that $E_0[x] \Downarrow_v$

t'_0 . By induction hypothesis, there exists a normal form t'_1 such that $E_1[x] \Downarrow_v t'_1$ and t'_0 and t'_1 are related by $\widehat{\mathcal{R}}$. By Lemma 10, we have $E_0[t_0] \rightarrow_v^* t'_0\{v'_0/x\}$ and $E_1[t_1] \rightarrow_v^* t'_1\{v'_1/x\}$. Suppose $E_0[t_0]$ reduces to $t'_0\{v'_0/x\}$ in more than one step. Then $t'_0\{v'_0/x\}$ evaluates to v_0 in strictly less than m steps. Because $t'_0\{v'_0/x\} \widehat{\mathcal{R}} t'_1\{v'_1/x\}$, by induction hypothesis, there exists v_1 such that $t'_1\{v'_1/x\} \Downarrow_v v_1$, and $v_0 z \widehat{\mathcal{R}} v_1 z$ for a fresh z . Because $E_1[t_1] \Downarrow_v v_1$, we have the required result. Suppose now that $E_0[t_0] = t'_0\{v'_0/x\}$. It is possible only if $t_0 = v'_0$ and $E_0[x] = E_0'[x v'_0]$. Then we have $t'_1 = E_1'[x v'_1]$ with $E_0'[x'] \widehat{\mathcal{R}} E_1'[x']$ for a fresh x' and $v'_0 \widehat{\mathcal{R}} v'_1$. Because $E_0'[v'_0 v'_0]$ evaluates to v_0 , v'_0 must be λ -abstraction $\lambda z.t''_0$. By a similar reasoning as in the case $t_0\{v_0/x\} \Downarrow_v v'_0$ (second sub-case), v'_1 is also a λ -abstraction $\lambda z.t''_1$. By Lemma 11, we have $t''_0 \widehat{\mathcal{R}} t''_1$. Therefore we have $E_0[t_0] \rightarrow_v E_0'[t''_0\{v''_0/z\}]$, $E_1[t_1] \rightarrow_v^* E_1'[t''_1\{v''_1/z\}]$ with $E_0'[t''_0\{v''_0/z\}] \widehat{\mathcal{R}} E_1'[t''_1\{v''_1/z\}]$. Because $E_1'[t''_1\{v''_1/z\}]$ evaluates to v_0 in less than $m - 1$ steps, by induction hypothesis, there exists v_1 such that $E_1'[t''_1\{v''_1/z\}] \Downarrow_v v_1$ and $v_0 z \widehat{\mathcal{R}} v_1 z$ for a fresh z . Because $E_1[t_1] \Downarrow_v v_1$, we have the required result.

The case $E_0[t_0] \Downarrow_v E_0'[Sk.t'_0]$ is similar. Suppose now that $E_0[t_0] \Downarrow_v F_0[y v_0]$. We have two possible cases; the case $t_0 \Downarrow_v F_0[y v_0]$ is easy using induction hypothesis. Suppose $t_0 \Downarrow_v v'_0$ and $E_0[v'_0] \Downarrow_v F_0[y v_0]$. By induction hypothesis, there exists v'_1 such that $t_1 \Downarrow_v v'_1$, and $v'_0 z \widehat{\mathcal{R}} v'_1 z$ for a fresh z . We have $v'_0 \widehat{\mathcal{R}} v'_1$ by Lemma 11. Because $E_0[v'_0] \Downarrow_v$, there exists a normal form t'_0 such that $E_0[x] \Downarrow_v t'_0$. By induction hypothesis, there exists a normal form t'_1 such that $E_1[x] \Downarrow_v t'_1$ and t'_0 and t'_1 are related by $\widehat{\mathcal{R}}$. By Lemma 10, we have $E_0[t_0] \rightarrow_v^* t'_0\{v'_0/x\}$ and $E_1[t_1] \rightarrow_v^* t'_1\{v'_1/x\}$. If $E_0[t_0]$ reduces to $t'_0\{v'_0/x\}$ in more than one step, then we proceed as in the case $E_0[t_0] \Downarrow_v v_0$. Otherwise, we have $t_0 = v'_0$, $E_0[x] = E_0'[x v'_0]$, $t'_1 = E_1'[x v'_1]$ with $E_0'[x'] \widehat{\mathcal{R}} E_1'[x']$ for a fresh x' , and $v'_0 \widehat{\mathcal{R}} v'_1$. If both v'_0 and v'_1 are λ -abstractions, then we proceed as in the case $E_0[t_0] \Downarrow_v v_0$. If they are both variables, then $v'_0 = v'_1 = y$, and the result holds. If v'_0 is a variable and v'_1 is a λ -abstraction, then we must have $v'_0 = y$, $F_0 = E_0''$, and $v_0 = v''_0$. Because we have $y z \widehat{\mathcal{R}} v_1 z$, by induction hypothesis, there exist E_1'' , v''_1 such that $v_1 z \Downarrow_v E_1''[y v''_1]$, $y' \widehat{\mathcal{R}} E_1''[y']$ for a fresh y' , and $z \widehat{\mathcal{R}} v''_1$. Consequently, we have $E_1[t_1] \Downarrow_v E_1'[E_1''\{v''_1/z\}][y v''_1\{v''_1/z\}]$. From $E_0'[x'] \widehat{\mathcal{R}} E_1'[x']$ and $y' \widehat{\mathcal{R}} E_1''[y']$, we deduce $E_0' \widehat{\mathcal{R}} E_1'[E_1''\{v''_1/z\}][]$. From $v'_0 \widehat{\mathcal{R}} v''_1$ and $z \widehat{\mathcal{R}} v''_1$, we deduce $v'_0 \widehat{\mathcal{R}} v''_1\{v''_1/z\}$. Consequently, we have the required result.

Assume $F_0[\langle E_0[t_0] \rangle] \widehat{\mathcal{R}} F_1[\langle E_1[t_1] \rangle]$ with $F_0[x] \widehat{\mathcal{R}} F_1[x]$, $E_0[x] \widehat{\mathcal{R}} E_1[x]$ (x fresh), and $t_0 \widehat{\mathcal{R}} t_1$. Note that $F_0[\langle E_0[t_0] \rangle]$ cannot evaluate to $E_0'[Sk.t'_0]$. Suppose $F_0[\langle E_0[t_0] \rangle] \Downarrow_v v_0$. We have several cases to consider.

- Suppose $t_0 \Downarrow_v v'_0$, $E_0[v'_0] \Downarrow_v v''_0$, and $F_0[\langle v''_0 \rangle] \Downarrow_v v_0$. By induction hypothesis, there exists v'_1 such that $t_1 \Downarrow_v v'_1$ and $v'_0 y \widehat{\mathcal{R}} v'_1 y$ for a fresh y . By Lemma 11, we have $v'_0 \widehat{\mathcal{R}} v'_1$. We have $E_1[t_1] \rightarrow_v^* E_1[v'_1]$ and $E_0[v'_0] \widehat{\mathcal{R}} E_1[v'_1]$. Because the evaluation $F_0[\langle v''_0 \rangle] \Downarrow_v v_0$ takes at least one step (corresponding to $\langle v''_0 \rangle \rightarrow_v v''_0$), we know that the evaluation $E_0[v'_0] \Downarrow_v v''_0$ is in $m - 1$

steps or less. Therefore, by induction hypothesis, there exists v_1'' such that $E_I[v_1'] \Downarrow_v v_1''$ and $v_0'' y \widehat{\mathcal{R}} v_1'' y$ for a fresh y . By Lemma 11, we have $v_0'' \widehat{\mathcal{R}} v_1''$. Because $F_0[\langle v_0'' \rangle] \Downarrow_v v_0$, there exists a normal form t_0' such that $F_0[x] \Downarrow_v t_0'$. By induction hypothesis, there exists a normal form t_1' such that $F_I[x] \Downarrow_v t_1'$ and t_0', t_1' are related by $\widehat{\mathcal{R}}$. We have $F_0[\langle v_0'' \rangle] \rightarrow_v^* t_0'\{v_0''/x\}$ and $F_I[\langle v_1'' \rangle] \rightarrow_v^* t_1'\{v_1''/x\}$, with $t_0'\{v_0''/x\} \widehat{\mathcal{R}} t_1'\{v_1''/x\}$. Because the reduction $F_0[\langle v_0'' \rangle] \rightarrow_v^* t_0'\{v_0''/x\}$ takes at least one step, we know that the evaluation $t_0'\{v_0''/x\} \Downarrow_v v_0$ takes $m-1$ steps or less. Therefore, by induction hypothesis, there exists v_1 such that $t_1'\{v_1''/x\} \Downarrow_v v_1$ and $v_0 y \widehat{\mathcal{R}} v_1 y$. Because $F_I[\langle E_I[t_1] \rangle] \Downarrow_v v_1$, we have the required result.

- Suppose $t_0 \Downarrow_v v_0'$, $E_0[v_0'] \Downarrow_v E_0'[\mathcal{S}k.t_0']$, and $F_0[\langle E_0'[\mathcal{S}k.t_0'] \rangle] \Downarrow_v v_0$. By induction hypothesis, there exists v_1' such that $t_1 \Downarrow_v v_1'$ and $v_0' y \widehat{\mathcal{R}} v_1' y$ for a fresh y . By Lemma 11, we have $v_0' \widehat{\mathcal{R}} v_1'$. We have $E_I[t_1] \rightarrow_v^* E_I[v_1']$ and $E_0[v_0'] \widehat{\mathcal{R}} E_I[v_1']$. Because the evaluation $F_0[\langle E_0'[\mathcal{S}k.t_0'] \rangle] \Downarrow_v v_0$ takes at least one step (corresponding to the capture of E_0' by shift), we know that the evaluation $E_0[v_0'] \Downarrow_v E_0'[\mathcal{S}k.t_0']$ is in $m-1$ steps or less. Therefore, by induction hypothesis, there exists E_1', t_1' such that $E_I[v_1'] \Downarrow_v E_1'[\mathcal{S}k.t_1']$, $\langle t_0' \rangle \widehat{\mathcal{R}} \langle t_1' \rangle$, and $E_0'[y] \widehat{\mathcal{R}} E_1'[y]$ for a fresh y . Hence, we have $\langle t_0' \{ \lambda y. \langle E_0'[y] \rangle / k \} \rangle \widehat{\mathcal{R}} \langle t_1' \{ \lambda y. \langle E_1'[y] \rangle / k \} \rangle$. Because $F_0[\langle E_0'[\mathcal{S}k.t_0'] \rangle]$ evaluates to v_0 , we must have $\langle E_0'[\mathcal{S}k.t_0'] \rangle \Downarrow_v v_0'$, $F_0[x] \Downarrow_v t_0'$ for some normal form t_0'' , and $t_0''\{v_0''/x\} \Downarrow_v v_0$. Because $\langle E_0'[\mathcal{S}k.t_0'] \rangle \rightarrow_v \langle t_0' \{ \lambda y. \langle E_0'[y] \rangle / k \} \rangle$, we know $\langle t_0' \{ \lambda y. \langle E_0'[y] \rangle / k \} \rangle$ evaluates to v_0'' in $m-1$ steps or less. Consequently, by induction hypothesis, there exists v_1'' such that $\langle t_1' \{ \lambda y. \langle E_1'[y] \rangle / k \} \rangle \Downarrow_v v_1''$ and $v_0'' z \widehat{\mathcal{R}} v_1'' z$ for a fresh z . Also, there exists a normal form t_1'' such that $F_I[x] \Downarrow_v t_1''$ and t_0'', t_1'' are related by $\widehat{\mathcal{R}}$. By Lemma 11, we have $v_0'' \widehat{\mathcal{R}} v_1''$, so we have $t_0''\{v_0''/x\} \widehat{\mathcal{R}} t_1''\{v_1''/x\}$. Because the reduction $F_0[\langle E_0'[\mathcal{S}k.t_0'] \rangle] \rightarrow_v^* t_0''\{v_0''/x\}$ takes at least one step, we know that the evaluation $t_0''\{v_0''/x\} \Downarrow_v v_0$ is in $m-1$ steps or less. Hence, by induction hypothesis, there exists v_1 such that $t_1''\{v_1''/x\} \Downarrow_v v_1$ and $v_0 y \widehat{\mathcal{R}} v_1 y$ for a fresh y . One can check that $F_I[\langle E_I[t_1] \rangle] \Downarrow_v v_1$ holds, therefore the required result holds.
- Suppose $t_0 \Downarrow_v E_0'[\mathcal{S}k.t_0]$ and $F_0[\langle E_0[E_0'[\mathcal{S}k.t_0]] \rangle] \Downarrow_v v_0$. This sub-case is similar to the previous one.

Suppose $F_0[\langle E_0[t_0] \rangle] \Downarrow_v F_0'[y v_0]$. There are five sub-cases to consider: three of them are similar to the sub-cases of $F_0[\langle E_0[t_0] \rangle] \Downarrow_v v_0$, and the remaining two are similar to the sub-cases of $E_0[t_0] \Downarrow_v F_0[y v_0]$ (namely $t_0 \Downarrow_v F_0''[y v_0]$ with $F_0' = F_0[\langle E_0[F_0''[\] \rangle]$, or $t_0 \Downarrow_v v_0'$, $E_0[v_0'] \Downarrow_v F_0''[y v_0]$ with $F_0' = F_0[\langle F_0''[\] \rangle]$). \square

Lemma 13 (Lemma 7 in the paper). *If \mathcal{R} is a refined bisimulation, then so is $\widetilde{\mathcal{R}}$.*

Because the proof is quite similar to the previous one, we sketch only the cases with the largest differences.

Proof (Sketch). Assume $t_0 \overset{\bullet}{\approx} t_0^1 \widetilde{\mathcal{R}} t_1^1 \overset{\bullet}{\approx} t_1$ and $t_0 \Downarrow_v t_0'$, where t_0' is a normal form. By bisimilarity, there exists t_0'' such that $t_0^1 \Downarrow_v t_0''$ and $t_0' \overset{\bullet}{\approx} t_0''$. By induction

hypothesis (on the definition of $\widetilde{\mathcal{R}}$), there exists t'_1 such that $t'_1 \Downarrow_v t''_1$ and $t'_0 \widetilde{\mathcal{R}} t'_1$. By bisimilarity, there exists t'_1 such that $t_1 \Downarrow_v t'_1$ and $t'_1 \overset{\bullet}{\approx} t'_1$. Finally, we have $t'_0 \overset{\bullet}{\approx} \widetilde{\mathcal{R}} \overset{\bullet}{\approx} t'_1$, and because $\overset{\bullet}{\approx} \widetilde{\mathcal{R}} \overset{\bullet}{\approx} \subseteq \widetilde{\mathcal{R}}$, we have the required result.

Assume we are in the case where $E_0[t_0] \widetilde{\mathcal{R}} E_1[t_1]$ with $E_0[y] \widetilde{\mathcal{R}} E_1[y]$ for a fresh y , and $t_0 \widetilde{\mathcal{R}} t_1$. Moreover, suppose $t_0 \Downarrow_v E_0'[\mathcal{S}k.t'_0]$. Then by induction hypothesis, we know that there exist E_1', t'_1 such that $t_1 \Downarrow_v E_1'[\mathcal{S}k.t'_1]$, and $\langle t'_0 \{ \lambda x. \langle k' E_0'[x] \} / k \} \rangle \widetilde{\mathcal{R}} \langle t'_1 \{ \lambda x. \langle k' E_1'[x] \} / k \} \rangle$ (*) for a fresh k' . Hence, we have $E_0[t_0] \Downarrow_v E_0[E_0'[\mathcal{S}k.t'_0]]$ and $t_1 \Downarrow_v E_1[E_1'[\mathcal{S}k.t'_1]]$, and we want to prove that $\langle t'_0 \{ \lambda x. \langle k' E_0[E_0'[x]] \} / k \} \rangle \widetilde{\mathcal{R}} \langle t'_1 \{ \lambda x. \langle k' E_1[E_1'[x]] \} / k \} \rangle$ holds. Because $E_0[y] \widetilde{\mathcal{R}} E_1[y]$, we have $\lambda y. k' E_0[y] \widetilde{\mathcal{R}} \lambda y. k' E_1[y]$ (**). Using (*) and (**), we have $\langle t'_0 \{ \lambda x. \langle (\lambda y. k' E_0[y]) E_0'[x] \} / k \} \rangle \widetilde{\mathcal{R}} \langle t'_1 \{ \lambda x. \langle (\lambda y. k' E_1[y]) E_1'[x] \} / k \} \rangle$, because $\widetilde{\mathcal{R}}$ is substitutive. By Lemma 6, we have

$$\begin{aligned} \langle t'_0 \{ \lambda x. \langle k' E_0[E_0'[x]] \} / k \} \rangle &\overset{\bullet}{\approx} \langle t'_0 \{ \lambda x. \langle (\lambda y. k' E_0[y]) E_0'[x] \} / k \} \rangle \\ \langle t'_1 \{ \lambda x. \langle k' E_1[E_1'[x]] \} / k \} \rangle &\overset{\bullet}{\approx} \langle t'_1 \{ \lambda x. \langle (\lambda y. k' E_1[y]) E_1'[x] \} / k \} \rangle, \end{aligned}$$

which means that $\langle t'_0 \{ \lambda x. \langle k' E_0[E_0'[x]] \} / k \} \rangle \overset{\bullet}{\approx} \widetilde{\mathcal{R}} \overset{\bullet}{\approx} \langle t'_1 \{ \lambda x. \langle k' E_1[E_1'[x]] \} / k \} \rangle$ holds. We have then the required result because $\overset{\bullet}{\approx} \widetilde{\mathcal{R}} \overset{\bullet}{\approx} \subseteq \widetilde{\mathcal{R}}$.

Assume $F_0[\langle E_0[t_0] \rangle] \widetilde{\mathcal{R}} F_1[\langle E_1[t_1] \rangle]$ with $F_0[x] \widetilde{\mathcal{R}} F_1[x]$, $E_0[x] \widetilde{\mathcal{R}} E_1[x]$ (x fresh), and $t_0 \widetilde{\mathcal{R}} t_1$. Moreover, suppose $t_0 \Downarrow_v v'_0$, $E_0[v'_0] \Downarrow_v E_0'[\mathcal{S}k.t'_0]$, and $F_0[\langle E_0'[\mathcal{S}k.t'_0] \rangle] \Downarrow_v v_0$. By induction hypothesis, there exists v'_1 such that $t_1 \Downarrow_v v'_1$ and $v'_0 y \widetilde{\mathcal{R}} v'_1 y$ for a fresh y . By Lemma 11, we have $v'_0 \widetilde{\mathcal{R}} v'_1$. We have $E_1[t_1] \rightarrow_v^* E_1[v'_1]$ and $E_0[v'_0] \widetilde{\mathcal{R}} E_1[v'_1]$. Because $F_0[\langle E_0'[\mathcal{S}k.t'_0] \rangle] \Downarrow_v v_0$ takes at least one step (corresponding to the capture of E_0' by shift), we know that the evaluation $E_0[v'_0] \Downarrow_v E_0'[\mathcal{S}k.t'_0]$ is in $m - 1$ steps or less. Therefore, by induction hypothesis, there exists E_1', t'_1 such that $E_1[v'_1] \Downarrow_v E_1'[\mathcal{S}k.t'_1]$ and $\langle t'_0 \{ \lambda y. \langle k' E_0'[y] \} / k \} \rangle \widetilde{\mathcal{R}} \langle t'_1 \{ \lambda y. \langle k' E_1'[y] \} / k \} \rangle$ for fresh y and k' . Using substitutivity of $\widetilde{\mathcal{R}}$, we obtain $\langle t'_0 \{ \lambda y. \langle (\lambda z. z) E_0'[y] \} / k \} \rangle \widetilde{\mathcal{R}} \langle t'_1 \{ \lambda y. \langle (\lambda z. z) E_1'[y] \} / k \} \rangle$, and by Lemma 6, we have

$$\begin{aligned} \langle t'_0 \{ \lambda y. \langle (\lambda z. z) E_0'[y] \} / k \} \rangle &\overset{\bullet}{\approx} \langle t'_0 \{ \lambda y. \langle E_0'[y] \} / k \} \rangle \\ \langle t'_1 \{ \lambda y. \langle (\lambda z. z) E_1'[y] \} / k \} \rangle &\overset{\bullet}{\approx} \langle t'_1 \{ \lambda y. \langle E_1'[y] \} / k \} \rangle. \end{aligned}$$

Consequently, we have $\langle t'_0 \{ \lambda y. \langle E_0'[y] \} / k \} \rangle \overset{\bullet}{\approx} \widetilde{\mathcal{R}} \overset{\bullet}{\approx} \langle t'_1 \{ \lambda y. \langle E_1'[y] \} / k \} \rangle$, and because $\overset{\bullet}{\approx} \widetilde{\mathcal{R}} \overset{\bullet}{\approx} \subseteq \widetilde{\mathcal{R}}$, we have $\langle t'_0 \{ \lambda y. \langle E_0'[y] \} / k \} \rangle \widetilde{\mathcal{R}} \langle t'_1 \{ \lambda y. \langle E_1'[y] \} / k \} \rangle$. From there, the proof is the same as in the same case of the proof for $\widehat{\mathcal{R}}$. \square