Bisimulations for Delimited-Control Operators

Dariusz Biernacki   Sergueï Lenglet

Abortive control operators

E.g., callcc in Scheme or SML/NJ

- Capture the entire rest of the computation (continuation)

Example

```
11+callcc(fn k => 100+k 0)
```

Captured continuation: $11 + [\,]$

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- When resumed, the captured continuation replaces the current continuation.

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- Capture the entire rest of the computation (continuation)
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Example

`11+callcc(fn k => 100+k 0)` ==>11

Captured continuation: $11 + [\,]$

Delimited-control operators

E.g., shift and reset (Danvy and Filinski, 90)

- Capture a prefix of the rest of the computation (delimited continuation)

Example

```
1+reset(fn () => 10 + shift (fn k => 100+k (k 0)))
```

Captured (composable) continuation: $10 + [\,]$

Delimited-control operators

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- Capture a prefix of the rest of the computation (delimited continuation)
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Example

```
1+reset(fn () => 10 + shift (fn k => 100+k (k 0)))
==>121
```

Captured (composable) continuation: $10 + [\,]$

Delimited-control operators: applications

Motivation

- Success-failure continuation model of backtracking
- More expressive than abortive control operators

Applications

- Non-deterministic programming
- Computational monads
- Partial evaluation
- Normalization by evaluation
- Linguistics
- Concurrency
- ...

Behavioral theory for λ-calculi

Contextual equivalence:

Definition

Terms $t_0 \mathcal{B} t_1$ contextually equivalent iff for all C, $C[t_0]$ and $C[t_1]$ have the same observable actions

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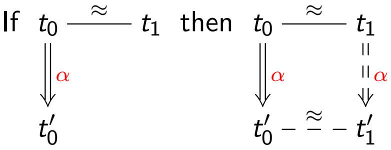
(Weak) bisimilarity:

Definition



The call-by-value λ-calculus with shift and reset

Syntax

Terms:	t	$::=$	$x \mid \lambda x.t \mid t\ t \mid \mathcal{S}k.t \mid \langle t \rangle$
Values:	v	$::=$	$\lambda x.t$
CBV contexts:	E	$::=$	$\square \mid v\ E \mid E\ t$

Call-by-value reduction:

$$\frac{}{(\lambda x.t)\ v \rightarrow_{\mathrm{v}} t\{v/x\}} \qquad \frac{t_1 \rightarrow_{\mathrm{v}} t_1'}{t_1\ t_2 \rightarrow_{\mathrm{v}} t_1'\ t_2} \qquad \frac{t \rightarrow_{\mathrm{v}} t'}{v\ t \rightarrow_{\mathrm{v}} v\ t'}$$

$$\frac{x \text{ fresh}}{\langle E[\mathcal{S}k.t] \rangle \rightarrow_{\mathrm{v}} \langle t\{\lambda x.\langle E[x] \rangle / k\} \rangle} \qquad \frac{}{\langle v \rangle \rightarrow_{\mathrm{v}} v} \qquad \frac{t \rightarrow_{\mathrm{v}} t'}{\langle t \rangle \rightarrow_{\mathrm{v}} \langle t' \rangle}$$

Captured (composable) context

Contextual equivalence

Evaluation of a closed term

- $t \Uparrow_{\mathrm{v}}$
- $t \Downarrow_{\mathrm{v}} v$
- $t \Downarrow_{\mathrm{v}} E[\mathcal{S}k.t']$

Definition (Contextual equivalence)

Let t_0, t_1 be closed terms. We have $t_0 \mathcal{B} t_1$ iff for all C,

- $C[t_0] \Downarrow_{\mathrm{v}} v_0$ implies $C[t_1] \Downarrow_{\mathrm{v}} v_1$;
- $C[t_0] \Downarrow_{\mathrm{v}} E_0[\mathcal{S}k.t'_0]$ implies $C[t_1] \Downarrow_{\mathrm{v}} E_1[\mathcal{S}k.t'_1]$.

and conversely for $C[t_1]$

Bisimilarity: actions α

$t \xrightarrow{\tau} t'$: internal action (= reduction)

$$\frac{}{(\lambda x.t)\ v \xrightarrow{\tau} t\{v/x\}} \qquad \frac{}{\langle v \rangle \xrightarrow{\tau} v} \qquad \frac{t \xrightarrow{\Box} t'}{\langle t \rangle \xrightarrow{\tau} t'}$$

$$\frac{t \xrightarrow{\tau} t'}{v\ t \xrightarrow{\tau} v\ t'} \qquad \frac{t_0 \xrightarrow{\tau} t_0'}{t_0\ t_1 \xrightarrow{\tau} t_0'\ t_1} \qquad \frac{t \xrightarrow{\tau} t'}{\langle t \rangle \xrightarrow{\tau} \langle t' \rangle}$$

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$v \xrightarrow{v_0} t'$: testing values (as in regular λ-calculus)

$$\frac{}{\lambda x.t \xrightarrow{v_0} t\{v_0/x\}}$$

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$v \xrightarrow{v_0} t'$: testing values (as in regular λ-calculus)

$$\frac{}{\lambda x.t \xrightarrow{v_0} t\{v_0/x\}}$$

$t \xrightarrow{E} t'$: context capture

- t is a stuck term $E_0[\mathcal{S}k.t_0]$
- t captures E and becomes t': $\langle E[t] \rangle \xrightarrow{\tau} t'$

Context capture

$$\frac{}{\langle v\,((\mathcal{S}k.t_0)\;t_1)\rangle \xrightarrow{\tau} \ \ldots}$$

Context capture

$$\frac{\overline{v\,((\mathcal{S}k.t_0)\;t_1) \xrightarrow{\square} \ \ldots}}{\langle v\,((\mathcal{S}k.t_0)\;t_1)\rangle \xrightarrow{\tau} \ \ldots}$$

Context capture

$$\dfrac{\dfrac{\overline{(\mathcal{S}k.t_0)\ t_1 \xrightarrow{v\ \square} \ \ldots}}{v\ ((\mathcal{S}k.t_0)\ t_1) \xrightarrow{\square} \ \ldots}}{\langle v\ ((\mathcal{S}k.t_0)\ t_1)\rangle \xrightarrow{\tau} \ \ldots}$$

Context capture

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathcal{S}k.t_0 \xrightarrow{v\,(\Box\; t_1)} \ldots}}{(\mathcal{S}k.t_0)\; t_1 \xrightarrow{v\,\Box} \ldots}}{v\,((\mathcal{S}k.t_0)\; t_1) \xrightarrow{\Box} \ldots}}{\langle v\,((\mathcal{S}k.t_0)\; t_1)\rangle \xrightarrow{\tau} \ldots}$$

Context capture

$$\dfrac{\dfrac{\dfrac{\dfrac{x \text{ fresh}}{\mathcal{S}k.t_0 \xrightarrow{v\ (\square\ t_1)} \langle t_0\{\lambda x.\langle v\ (x\ t_1)\rangle / k\}\rangle}}{(\mathcal{S}k.t_0)\ t_1 \xrightarrow{v\ \square} \ldots}}{v\ ((\mathcal{S}k.t_0)\ t_1) \xrightarrow{\square} \ldots}}{\langle v\ ((\mathcal{S}k.t_0)\ t_1)\rangle \xrightarrow{\tau} \ldots}$$

Context capture

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Applicative bisimilarity

Weak transitions

$$\begin{aligned} \overset{\tau}{\Rightarrow} \quad &\overset{\Delta}{=} \quad (\overset{\tau}{\rightarrow})^* \\ \overset{\alpha}{\Rightarrow} \quad &\overset{\Delta}{=} \quad \overset{\tau}{\Rightarrow}\overset{\alpha}{\rightarrow} \ \text{ if } \alpha \neq \tau \end{aligned}$$

Definition (Applicative bisimilarity)

If $\begin{array}{ccc} t_0 & \overset{\approx}{\rule{2em}{0.4pt}} & t_1 \\ \Downarrow{\scriptstyle\alpha} & & \\ t_0' & & \end{array}$ $(\alpha \neq \tau)$ then $\begin{array}{ccc} t_0 & \overset{\approx}{\rule{2em}{0.4pt}} & t_1 \\ \Downarrow{\scriptstyle\alpha} & & \Downarrow{\scriptstyle\alpha} \\ t_0' & \overset{\approx}{- - -} & t_1' \end{array}$

Theorem

We have $\mathcal{B} = \approx$

Equivalences of terms

$$\mathcal{S}k.\langle t\rangle \quad \approx \quad \mathcal{S}k.t$$

Because

$\mathcal{S}k.\langle t\rangle \xrightarrow{E} \langle\langle t\{\lambda x.\langle E[x]\rangle/k\}\rangle\rangle$

$\mathcal{S}k.t \xrightarrow{E} \langle t\{\lambda x.\langle E[x]\rangle/k\}\rangle$

and $\forall t', \langle\langle t'\rangle\rangle \approx \langle t'\rangle$

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and $\forall t', \langle\langle t'\rangle\rangle \approx \langle t'\rangle$

$$(\lambda x.E[x])\; t \quad \approx \quad E[t]$$

x fresh

More complex ...

CPS equivalence vs bisimilarity: β_Ω axiom

$$\mathcal{R}_1 = \{((\lambda x.E[x])\ t', E[t'])\}$$

$$(\lambda x.E[x])\ t' \quad \mathcal{R}_1 \quad E[t']$$

CPS equivalence vs bisimilarity: β_Ω axiom

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$$(\lambda x.E[x])\; \mathcal{S}k.t \qquad \mathcal{R}_1 \qquad E[\mathcal{S}k.t]$$

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$$(\lambda x.E[x])\, \mathcal{S}k.t \qquad \mathcal{R}_1 \qquad E[\mathcal{S}k.t]$$

$$\downarrow E_0$$

$$\langle t\{\lambda y.\langle E_0[(\lambda x.E[x])\, y]\rangle / k\}\rangle$$

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CPS equivalence vs bisimilarity: β_Ω axiom

$$\mathcal{R}_1 = \{((\lambda x.E[x])\, t', E[t'])\}$$

$$\mathcal{R}_2 = \{(t''\sigma_1, t''\sigma_2)\}$$

$$\begin{array}{ccc}
(\lambda x.E[x])\, \mathcal{S}k.t & \mathcal{R}_1 & E[\mathcal{S}k.t] \\
\big\downarrow E_0 & & \big\downarrow E_0 \\
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F\sigma_1[\lambda y.\langle E_0[(\lambda x.E[x])\ y]\rangle\ v\sigma_1] & \mathcal{R}_2 & F\sigma_2[\lambda y.\langle E_0[E[y]]\rangle\ v\sigma_2]
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CPS equivalence vs bisimilarity: β_Ω axiom

$$
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$$

$$
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\downarrow E_0 & & \downarrow E_0 \\
F\sigma_1[\lambda y.\langle E_0[(\lambda x.E[x])\ y]\rangle\ v\sigma_1] & \mathcal{R}_2 & F\sigma_2[\lambda y.\langle E_0[E[y]]\rangle\ v\sigma_2] \\
\downarrow \tau & & \downarrow \tau \\
F\sigma_1[\langle E_0[(\lambda x.E[x])\ v\sigma_1]\rangle] & & F\sigma_2[\langle E_0[E[v\sigma_2]]\rangle]
\end{array}
$$

CPS equivalence vs bisimilarity: β_Ω axiom

$$\mathcal{R}_1 = \{(F\sigma_1^1 \ldots \sigma_1^n[(\lambda x.E[x])\; t'\sigma_1^1 \ldots \sigma_1^n], F\sigma_2^1 \ldots \sigma_2^n[E[t'\sigma_2^1 \ldots \sigma_2^n]])\}$$

$$\mathcal{R}_2 = \{(t''\sigma_1, t''\sigma_2)\}$$

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F\sigma_1[\lambda y.\langle E_0[(\lambda x.E[x])\; y]\rangle\; v\sigma_1] & \mathcal{R}_2 & F\sigma_2[\lambda y.\langle E_0[E[y]]\rangle\; v\sigma_2] \\
\downarrow \tau & & \downarrow \tau \\
F\sigma_1[\langle E_0[(\lambda x.E[x])\; v\sigma_1]\rangle] & \mathcal{R}_1 & F\sigma_2[\langle E_0[E[v\sigma_2]]\rangle]
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$$\mathcal{R}_2 = \{(t''\sigma_1^1 \ldots \sigma_1^n, t''\sigma_2^1 \ldots \sigma_2^n)\}$$

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\downarrow \tau & & \downarrow \tau \\
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\end{array}$$

Applicative bisimilarity: summary

Definition (Applicative bisimilarity)

If $\begin{array}{ccc} t_0 & \overset{\approx}{\longrightarrow\!\!\!\!\!-} & t_1 \\ \Big\Downarrow{\scriptstyle\alpha} & & \\ t_0' & & \end{array}$ $(\alpha \neq \tau)$ then $\begin{array}{ccc} t_0 & \overset{\approx}{\longrightarrow\!\!\!\!\!-} & t_1 \\ \Big\Downarrow{\scriptstyle\alpha} & & \Big\Downarrow{\scriptstyle\alpha} \\ t_0' & \overset{\approx}{- -} & t_1' \end{array}$

- Sound and complete w.r.t. contextual equivalence
- Proofs can be complex: still a quantification on values v, and on contexts E

Can we do better?

Normal form bisimilarity (open bisimilarity)

Principle (Sangiorgi, Lassen):

- Reduce (open) terms to normal forms
- Decompose normal forms into bisimilar subterms

Example: λ-calculus

Definition

If $t_0 \approx_{\mathrm{NF}} t_1$ then:

- If $t_0 \Downarrow_{\mathrm{v}} x$, then $t_1 \Downarrow_{\mathrm{v}} x$
- If $t_0 \Downarrow_{\mathrm{v}} \lambda x.t_0'$, then $t_1 \Downarrow_{\mathrm{v}} \lambda x.t_1'$ and $t_0' \approx_{\mathrm{NF}} t_1'$
- If $t_0 \Downarrow_{\mathrm{v}} E_0[x\ v_0]$, then $t_1 \Downarrow_{\mathrm{v}} E_1[x\ v_1]$, $v_0 \approx_{\mathrm{NF}} v_1$ and $E_0[y] \approx_{\mathrm{NF}} E_1[y]$ for a fresh y

Evaluation of open terms

Terms:	t	$::=$	$x \mid \lambda x.t \mid t\ t \mid \mathcal{S}k.t \mid \langle t \rangle$
Values:	v	$::=$	$\lambda x.t \mid x$
CBV contexts:	E	$::=$	$\square \mid v\ E \mid E\ t$
Evaluation contexts:	F	$::=$	$\square \mid v\ F \mid F\ t \mid \langle F \rangle$

Evaluation of an open term

- $t \Uparrow_{\mathrm{v}}$
- $t \Downarrow_{\mathrm{v}} v$
- $t \Downarrow_{\mathrm{v}} E[\mathcal{S}k.t']$
- $t \Downarrow_{\mathrm{v}} F[x\ v]$

Normal form bisimilarity for delimited control

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- If $t_0 \Downarrow_{\mathrm{v}} F_0[x\ v_0]$, then $t_1 \Downarrow_{\mathrm{v}} F_1[x\ v_1]$, $v_0 \approx_{\mathrm{NF}} v_1$ and $F_0 \approx_{\mathrm{NF}} F_1$.

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- If $t_0 \Downarrow_{\mathrm{v}} F_0[x\, v_0]$, then $t_1 \Downarrow_{\mathrm{v}} F_1[x\, v_1]$, $v_0 \approx_{\mathrm{NF}} v_1$ and $F_0 \approx_{\mathrm{NF}} F_1$.
- If $t_0 \Downarrow_{\mathrm{v}} E_0[\mathcal{S}k.t_0']$ then $t_1 \Downarrow_{\mathrm{v}} E_1[\mathcal{S}k.t_1']$, $E_0[x] \approx_{\mathrm{NF}} E_1[x]$ for a fresh x and $\langle t_0' \rangle \approx_{\mathrm{NF}} \langle t_1' \rangle$

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- If $t_0 \Downarrow_{\mathrm{v}} F_0[x\ v_0]$, then $t_1 \Downarrow_{\mathrm{v}} F_1[x\ v_1]$, $v_0 \approx_{\mathrm{NF}} v_1$ and $F_0 \approx_{\mathrm{NF}} F_1$.
- If $t_0 \Downarrow_{\mathrm{v}} E_0[\mathcal{S}k.t_0']$ then $t_1 \Downarrow_{\mathrm{v}} E_1[\mathcal{S}k.t_1']$, $E_0[x] \approx_{\mathrm{NF}} E_1[x]$ for a fresh x and $\langle t_0' \rangle \approx_{\mathrm{NF}} \langle t_1' \rangle$

Theorem

We have $\approx_{\mathrm{NF}} \subsetneq \mathcal{B}$.

Example: β_Ω axiom

$$\mathcal{R}_1 = \{((\lambda x.E[x])\ t', E[t'])\}$$

- $(\lambda x.E[x])\,\mathcal{S}k.t\ \mathcal{R}_1\ E[\mathcal{S}k.t]$

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  For $\mathcal{R}_1$ to be a normal form bisimulation, we need
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  - $\langle t \rangle\ \mathcal{R}_1\ \langle t \rangle$

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  - $(\lambda x.E[x])\,y\ \mathcal{R}_1\ E[y]$ for a fresh y True
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Example: β_Ω axiom

$$\mathcal{R}_1 = \{((\lambda x.E[x])\ t', E[t'])\} \cup \{(t, t)\}$$

- $(\lambda x.E[x])\, \mathcal{S}k.t\ \mathcal{R}_1\ E[\mathcal{S}k.t]$
  For $\mathcal{R}_1$ to be a normal form bisimulation, we need
  - $(\lambda x.E[x])\, y\ \mathcal{R}_1\ E[y]$ for a fresh y True
  - $\langle t \rangle\ \mathcal{R}_1\ \langle t \rangle$ Easy change

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  - $(\lambda x.E[x])\, y\ \mathcal{R}_1\ E[y]$ for a fresh y True
  - $\langle t \rangle\ \mathcal{R}_1\ \langle t \rangle$ Easy change
- That's it

Conclusions so far

- Applicative bisimilarity
  - Equivalence proofs can be quite involved
  - Complete
  - Up-to techniques: open problem
- Normal form bisimilarity
  - Equivalence proofs are quite simple
  - Not complete
  - Allow for up-to techniques

Environmental bisimulation

Definition

A relation $\mathcal{X}$ is an environmental bisimulation if

1. $t_0 \; \mathcal{X}_{\mathcal{E}} \; t_1$ implies:
   1.1 if $t_0 \rightarrow_{\mathrm{v}} t_0'$, then $t_1 \rightarrow_{\mathrm{v}}^{*} t_1'$ and $t_0' \; \mathcal{X}_{\mathcal{E}} \; t_1'$;
   1.2 if $t_0 = v_0$, then $t_1 \rightarrow_{\mathrm{v}}^{*} v_1$ and $\mathcal{E} \cup \{(v_0, v_1)\} \in \mathcal{X}$;

   1.4 the converse of the above conditions on t_1;
2. $\mathcal{E} \in \mathcal{X}$ implies:
   2.1 if $\lambda x.t_0 \; \mathcal{E} \; \lambda x.t_1$ and $v_0 \; \widehat{\mathcal{E}} \; v_1$, then $t_0\{v_0/x\} \; \mathcal{X}_{\mathcal{E}} \; t_1\{v_1/x\}$;

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   1.3 if t_0 is stuck, then $t_1 \rightarrow_{\mathrm{v}}^{*} t_1'$ with t_1' stuck, and $\mathcal{E} \cup \{(t_0, t_1')\} \in \mathcal{X}$;
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   2.2 if $E_0[\mathcal{S}k.t_0] \ \mathcal{E} \ E_1[\mathcal{S}k.t_1]$ and ${E_0}' \ \widetilde{\mathcal{E}} \ {E_1}'$, then $\langle t_0\{\lambda x.\langle {E_0}'[E_0[x]]\rangle/k\}\rangle \ \mathcal{X}_{\mathcal{E}} \ \langle t_1\{\lambda x.\langle {E_1}'[E_1[x]]\rangle/k\}\rangle$ for a fresh x.

Up-to context

Definition

A relation $\mathcal{X}$ is an environmental bisimulation up to context if

1. $t_0 \; \mathcal{X}_{\mathcal{E}} \; t_1$ implies:
   1.1 if $t_0 \rightarrow_{\mathrm{v}} t_0'$, then $t_1 \rightarrow_{\mathrm{v}}^{*} t_1'$ and $t_0' \; \overline{\mathcal{X}_{\mathcal{E}}} \; t_1'$;
   1.2 . . .
2. $\mathcal{E} \in \mathcal{X}$ implies:
   2.1 if $\lambda x.t_0 \; \mathcal{E} \; \lambda x.t_1$ and $v_0 \; \widehat{\mathcal{E}} \; v_1$, then $t_0\{v_0/x\} \; \overline{\mathcal{X}_{\mathcal{E}}} \; t_1\{v_1/x\}$;
   2.2 if $E_0[\mathcal{S}k.t_0] \; \mathcal{E} \; E_1[\mathcal{S}k.t_1]$ and $E_0' \; \widetilde{\mathcal{E}} \; E_1'$, then $\langle t_0\{\lambda x.\langle E_0'[E_0[x]]\rangle/k\}\rangle \; \overline{\mathcal{X}_{\mathcal{E}}} \; \langle t_1\{\lambda x.\langle E_1'[E_1[x]]\rangle/k\}\rangle$ for a fresh x.

where $t_0 \; \overline{\mathcal{X}_{\mathcal{E}}} \; t_1$ if

- either $t_0 = F_0[t_0']$, $t_1 = F_1[t_1']$, $t_0' \; \mathcal{X}_{\mathcal{E}} \; t_1'$, and $F_0 \; \widetilde{\mathcal{E}} \; F_1$;
- or $t_0 \; \widehat{\mathcal{E}} \; t_1$.

Is it helpful?

- building $\mathcal{X}$ for $E[t] \simeq (\lambda x.E[x])\ t$

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- problematic case $t = E_0[\mathcal{S}k.t_0]$
- we add $(\lambda x.E[x])\ E_0[\mathcal{S}k.t_0]$ and $E[E_0[\mathcal{S}k.t_0]]$ to an environment $\mathcal{E}$ of $\mathcal{X}$

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- for all $E_1\ \widetilde{\mathcal{E}}\ E_2$, $\langle t'\{\lambda y.\langle E_1[(\lambda x.E[x])\ E_0[y]]\rangle/k\}\rangle\ \mathcal{X}_{\mathcal{E}}$ $\langle t'\{\lambda y.\langle E_2[E[E_0[y]]]\rangle/k\}\rangle$

Is it helpful?

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- for all $E_1\ \widetilde{\mathcal{E}}\ E_2$, $\langle t'\{\lambda y.\langle E_1[(\lambda x.E[x])\ E_0[y]]\rangle/k\}\rangle\ \mathcal{X}_{\mathcal{E}}\ \langle t'\{\lambda y.\langle E_2[E[E_0[y]]]\rangle/k\}\rangle$
- but:
  - $(\lambda x.E[x])\ E_0[y]$ and $E[E_0[y]]$ are open
  - t' can be any term
- we have $\langle t'\{\lambda y.\langle E_1[(\lambda x.E[x])\ E_0[y]]\rangle/k\}\rangle\ \widehat{\mathcal{X}_{\mathcal{E}}{}^{\circ}}\ \langle t'\{\lambda y.\langle E_2[E[E_0[y]]]\rangle/k\}\rangle$ instead of $\overline{\mathcal{X}_{\mathcal{E}}}$

Up-to context

Fix (?) for "any context issue"

- if $E_0[\mathcal{S}k.t_0] \; \mathcal{E} \; E_1[\mathcal{S}k.t_1]$ and $E_0' \; \widetilde{\mathcal{X}_{\mathcal{E}}} \; E_1'$, then $\langle t_0\{\lambda x.\langle E_0'[E_0[x]]\rangle/k\}\rangle \; \widehat{\mathcal{X}_{\mathcal{E}}} \; \langle t_1\{\lambda x.\langle E_1'[E_1[x]]\rangle/k\}\rangle$ for a fresh x.

Up-to context

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Fix for open terms: ???????

- $(\lambda x.E[x])\ E_0[y]\ \mathcal{X}_{\mathcal{E}}{}^{\circ}\ E[E_0[y]]$ implies $(\lambda x.E[x])\ E_0[v_0]\ \mathcal{X}_{\mathcal{E}}\ E[E_0[v_1]]$ for all $v_0\ \widehat{\mathcal{E}}\ v_1$.
- which implies $(\lambda x.E[x])\ t_0\ \mathcal{X}_{\mathcal{E}}\ E[t_1]$ with $t_0\ \widehat{\mathcal{E}}\ t_1$

Relaxed vs original semantics

- Relations so far defined for the relaxed semantics
  - Not complete w.r.t. CPS
  - $\mathcal{S}k.k\ t$ not equivalent to t ($k \notin \mathsf{fv}(t)$)
- Original semantics: terms are executed within a reset $\langle t \rangle$ (programs)
- Consequence: evaluation to a stuck term impossible

Corresponding contextual equivalence:

Definition

$t_0 \mathrel{\dot{\approx}_c} t_1$ if for all C, $\langle C[t_0] \rangle \Downarrow_{\mathrm{v}}$ implies $\langle C[t_1] \rangle \Downarrow_{\mathrm{v}}$, and conversely for $\langle C[t_1] \rangle$.

Environmental bisimilarity for the original semantics

Definition

A relation $\mathcal{X}$ is an environmental bisimulation for programs if

1. if $t_0 \; \mathcal{X}_{\mathcal{E}} \; t_1$ and t_0 and t_1 are not both programs, then for all $E_0 \; \widetilde{\mathcal{E}} \; E_1$, we have $\langle E_0[t_0] \rangle \; \mathcal{X}_{\mathcal{E}} \; \langle E_1[t_1] \rangle$;
2. if $p_0 \; \mathcal{X}_{\mathcal{E}} \; p_1$
   - 2.1 if $p_0 \rightarrow_{\mathrm{v}} p_0'$, then $p_1 \rightarrow_{\mathrm{v}}^{*} p_1'$ and $p_0' \; \mathcal{X}_{\mathcal{E}} \; p_1'$;
   - 2.2 if $p_0 \rightarrow_{\mathrm{v}} v_0$, then $p_1 \rightarrow_{\mathrm{v}}^{*} v_1$, and $\{(v_0, v_1)\} \cup \mathcal{E} \in \mathcal{X}$;
   - 2.3 the converse of the above conditions on p_1;
3. for all $\mathcal{E} \in \mathcal{X}$, if $\lambda x.t_0 \; \mathcal{E} \; \lambda x.t_1$ and $v_0 \; \widehat{\mathcal{E}} \; v_1$, then $t_0\{v_0/x\} \; \mathcal{X}_{\mathcal{E}} \; t_1\{v_1/x\}$.

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Stuff

- Soundness proof requires some tweaking w.r.t. λ-calculus
- Howe's method seems to fail with corresponding applicative bisimilarity
- Complete w.r.t. CPS
- Differences between the two semantics:

Lemma

- *We have* $\Omega \stackrel{\cdot}{\simeq} \mathcal{S}k.\Omega$.
- *If* $k \notin \mathsf{fv}(t)$, *then* $t \stackrel{\cdot}{\simeq} \mathcal{S}k.k\ t$.
- *If* $k \notin \mathsf{fv}(t_1)$ *and* $x \notin \mathsf{fv}(E)$, *then we have* $(\lambda x.E[\mathcal{S}k.t_0])\ t_1 \stackrel{\cdot}{\simeq} E[\mathcal{S}k.((\lambda x.t_0)\ t_1)]$.

Conclusion

Relaxed semantics:

- Applicative
- Normal form
- Environmental

Original semantics:

- environmental

Future work: abortive control ($\lambda\mu$)

- conjecture: environmental