Coalgebras over enriched categories

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- Coalgebras provide a unifying framework for a wide range of state-based systems.
- Examples: infinite labelled trees, finite automata, streams, transition systems, Kripke structures.
- Given an endofunctor $T : C \to C$, a T-coalgebra is a C-morphism

$$\xi : X \to TX$$

- T induces a notion of behaviour equivalence that generalizes the bisimilarity defined for each specific system.

Moss' logic

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$$(\mathcal{P} + T)(\mathcal{L}) \to \mathcal{L}$$

- This means that $\mathcal{L}$ is closed under ∇, if $\gamma \in T(\mathcal{L})$ then $\nabla\gamma \in \mathcal{L}$.

Example: ∇ for the powerset functor P

If X is a Kripke frame with the accessibility relation R and γ is a set of formulas then
$\Vdash (x, \nabla\gamma)$ when

- for all $\alpha \in \gamma$ there is $y \in R(x)$ s.t. $\Vdash (y, \alpha)$ and
- for all $y \in R(x)$ there exists $\alpha \in \gamma$ s.t. $\Vdash (y, \alpha)$.

In the standard modal language we have

$$\nabla\gamma = \Box \bigvee \gamma \wedge \bigwedge \Diamond\gamma$$

but also $\Box$ and $\Diamond$ can be defined in terms of ∇.

The semantics of ∇

The semantics of the logic w.r.t. a coalgebra $\xi : X \to TX$ and a state x in X is described via a relation

$$\Vdash_\xi \subseteq X \times \mathcal{L}$$

The semantics of the operator ∇ is then given using the relation lifting $\overline{T}$ via the inductive clause

$$\Vdash (x, \nabla\gamma) \;\Leftrightarrow\; \overline{T}(\Vdash)(\xi(x), \gamma).$$

Thus the next diagram commutes in the category Rel of sets and relations

$$\begin{array}{ccc} T\mathcal{L} & \xrightarrow{\nabla} & \mathcal{L} \\ {\scriptstyle \overline{T}(\Vdash)}\big\downarrow\!\!\!\!\!\!-\; & & \;\big\downarrow\!\!\!\!\!\!-\,{\scriptstyle \Vdash} \\ TX & \longrightarrow\!\!\!\!\!\!\!\!\!/\;\;\; & X \end{array}$$

We need T to preserve weak pullbacks. Under this assumption Moss also showed that ∇ is invariant under bisimilarity.

Classical result on relation lifting

Theorem

For a functor $T : \mathsf{Set} \to \mathsf{Set}$ *the following are equivalent:*

- *There is a monotone functor* $\overline{T} : \mathsf{Rel}(\mathsf{Set}) \to \mathsf{Rel}(\mathsf{Set})$ *such that the square*

$$\begin{array}{ccc} \mathsf{Rel}(\mathsf{Set}) & \xrightarrow{\ \overline{T}\ } & \mathsf{Rel}(\mathsf{Set}) \\ {\scriptstyle (-)_\diamond}\uparrow & & \uparrow{\scriptstyle (-)_\diamond} \\ \mathsf{Set} & \xrightarrow[T]{} & \mathsf{Set} \end{array}$$

  commutes.
- T *preserves weak pullbacks.*

Moving to the many-valued setting

The satisfaction relation $\Vdash: X \times \mathcal{L} \to 2$ could take more values, say in $[0, 1]$.

The idea is to use 'relations' in an enriched setting, so we will consider

$$\Vdash: X^{op} \otimes \mathcal{L} \to \mathscr{V}$$

where $\mathscr{V}$ is a commutative quantale.
We will enrich all our categories, functors, etc. in a (co)complete symmetric monoidal closed base category $\mathscr{V}$.

Commutative Quantales

A *commutative quantale*

$$\mathscr{V} = (\mathscr{V}_o, \otimes, I, [-,-])$$

is a symmetric monoidal structure such that

- $\mathscr{V}_o$ is a complete lattice with the lattice order written as $\leq$, the symbol $\perp$ denotes the least element and $\top$ the greatest element of $\mathscr{V}_o$.
- $\otimes$ is a symmetric monoidal structure on $\mathscr{V}_o$ with a unit element I.
- The closed structure (the internal hom) of $\mathscr{V}_o$ is denoted by $[x, y]$. Hence we have adjunction relations

  $$x \otimes y \leq z \quad \text{iff} \quad y \leq [x, z]$$

  for every x, y, z in $\mathscr{V}_o$.

Examples of quantales

1. The unit interval $[0;1]$ with the usual order and the Łukasiewicz tensor $x \otimes y = \max\{x + y - 1, 0\}$.
   The internal hom is given by $[x, y] \;=\;$ if $x \leq y$ then 1 else $1 - x + y$.
2. The unit interval $[0;1]$ with the usual order and the Gödel tensor $x \otimes y = \min\{x, y\}$.
   The internal hom is given by $[x, y] \;=\;$ if $x \leq y$ then 1 else y.
3. The unit interval $[0;1]$ with the usual order and the product tensor $x \otimes y = x \cdot y$.
   The internal hom is given by $[x, y] \;=\;$ if $x \leq y$ then 1 else $\frac{y}{x}$.

More examples

1. $\mathscr{V}_o$ is the two-element chain 2, i.e., there are two objects 0 and 1 with $0 \leq 1$.
   The tensor in 2 is the meet and the internal hom is implication.

2. $\mathscr{V}_o$ is the unit interval $[0;1]$ with $\leq$ being the reversed order $\geq_{\mathbb{R}}$ of the real numbers.
   The unit I is 0 and $x \otimes y = \max\{x, y\}$ where the maximum is taken w.r.t. the usual order $\leq_{\mathbb{R}}$.
   The internal hom is given by $[0,1](x,y) = \text{ if } x \geq_{\mathbb{R}} y \text{ then } 0 \text{ else } y$.

3. $\mathscr{V}_o$ is the interval $[0;\infty]$ with $\leq$ being the reversed order $\geq_{\mathbb{R}}$ of the reals.
   Extend the usual addition of nonnegative reals by putting $x + \infty = \infty + x = \infty$, for every $x \in [0;\infty]$ and let $x \otimes y = x + y$, the unit I being 0.
   The internal hom is given by truncated substraction $[0,1](x,y) = y \dot{-} x = \text{ if } x \geq_{\mathbb{R}} y \text{ then } 0 \text{ else } y - x$.

$\mathscr{V}$-categories

Definition

A category $\mathscr{A}$ *enriched* in $\mathscr{V}$ (or, a *$\mathscr{V}$-category*) consists of:

1. A class of *objects* denoted by $a, b, \ldots$
2. For every pair a, b of objects a *hom-object* $\mathscr{A}(a, b)$ in $\mathscr{V}_o$.

such that

1. For every object a there is an inequality
$$I \leq \mathscr{A}(a, a)$$
witnessing the "choice of the identity morphism on a".
2. For every triple a, b, c of objects there is an inequality
$$\mathscr{A}(b, c) \otimes \mathscr{A}(a, b) \leq \mathscr{A}(a, c)$$
witnessing "the composition of morphisms".

Examples

1. When $\mathscr{V} = 2$, a $\mathscr{V}$-category is a preorder.
2. When $\mathscr{V}$ is $[0, 1]$ with the reversed order, a $\mathscr{V}$-category $\mathscr{A}$ is a *generalised ultrametric space*: the hom-object $\mathscr{A}(a, b)$ is the “distance” of a and b
3. When $\mathscr{V}$ is $[0, \infty]$ with the reversed order, a $\mathscr{V}$-category $\mathscr{A}$ is a *generalised metric space*: the hom-object $\mathscr{A}(a, b)$ is the “distance” of a and b

$\mathscr{V}$-functors

Definition

Given $\mathscr{V}$-categories $\mathscr{A}$, $\mathscr{B}$, a *$\mathscr{V}$-functor* $f : \mathscr{A} \to \mathscr{B}$ is given by the following data:

1. An *object assignment*: for every object a in $\mathscr{A}$, there is a unique object fa in $\mathscr{B}$.
2. An *action on hom-objects*: for every pair a, a' of objects of $\mathscr{A}$ there is an inequality

$$\mathscr{A}(a, a') \leq \mathscr{B}(fa, fa')$$

in $\mathscr{V}_o$.

Relations in enriched setting

For a general base category $\mathscr{V}$, a "relation"

$$R : \mathscr{A} \longrightarrow\!\!\!\!\!\!\!+\;\;\; \mathscr{B}$$

from a $\mathscr{V}$-category $\mathscr{A}$ to a $\mathscr{V}$-category $\mathscr{B}$ is a $\mathscr{V}$-functor of the form

$$R : \mathscr{B}^{op} \otimes \mathscr{A} \to \mathscr{V}$$

called a *module* and
Given modules

$$R : \mathscr{A} \longrightarrow\!\!\!\!\!\!\!+\;\;\; \mathscr{B} \qquad S : \mathscr{B} \longrightarrow\!\!\!\!\!\!\!+\;\;\; \mathscr{C}$$

we define their *composite*

$$S \cdot R : \mathscr{A} \longrightarrow\!\!\!\!\!\!\!+\;\;\; \mathscr{C}$$

to be the functor with values

$$S \cdot R(c, a) = \bigvee_b S(c, b) \otimes R(b, a)$$

for all c and a.
By $\mathscr{V}$-mod we denote the 2-category of $\mathscr{V}$-modules (= "relations")

A 2-functor $(-)_\diamond : \mathscr{V}\text{-cat} \to \mathscr{V}\text{-mod}$

Definition

Given $f : \mathscr{A} \to \mathscr{B}$ in $\mathscr{V}$-cat, the module $f_\diamond : \mathscr{A} \longrightarrow\!\!\!\!\!\!\!\!\mapsto \mathscr{B}$ given by

$$f_\diamond(b, a) = \mathscr{B}(b, fa)$$

is called the *graph of* f.

every module $f_\diamond$ is a left adjoint in $\mathscr{V}$-mod, having the module

$$f^\diamond(a, b) = \mathscr{B}(fa, b)$$

as a right adjoint.

When can we lift a $\mathscr{V}$-cat functor to $\mathscr{V}$-mod?

A *relation lifting* of a 2-functor $T : \mathscr{V}\text{-cat} \to \mathscr{V}\text{-cat}$ is a 2-functor $\overline{T} : \mathscr{V}\text{-mod} \to \mathscr{V}\text{-mod}$, making the square

$$\begin{array}{ccc}
\mathscr{V}\text{-mod} & \xrightarrow{\;\overline{T}\;} & \mathscr{V}\text{-mod} \\
{\scriptstyle (-)_\diamond}\uparrow & & \uparrow{\scriptstyle (-)_\diamond} \\
\mathscr{V}\text{-cat} & \xrightarrow[\;T\;]{} & \mathscr{V}\text{-cat}
\end{array}$$

commutative.

Exact squares

Definition

Call a lax square

$$\begin{array}{ccc} \mathscr{P} & \xrightarrow{p_1} & \mathscr{B} \\ {\scriptstyle p_0}\downarrow & \nearrow & \downarrow{\scriptstyle g} \\ \mathscr{A} & \xrightarrow[f]{} & \mathscr{C} \end{array}$$

in $\mathscr{V}$-cat *exact*, if the equality

$$\mathscr{C}(fa, gb) = \bigvee_{w} \mathscr{A}(a, p_0 w) \otimes \mathscr{B}(p_1 w, b)$$

holds, naturally in a and b.

The main result

Theorem

For $T : \mathscr{V}\text{-cat} \to \mathscr{V}\text{-cat}$*, the following are equivalent:*

1. *There exists* $\overline{T} : \mathscr{V}\text{-mod} \to \mathscr{V}\text{-mod}$ *such that the following square*

$$\begin{array}{ccc} \mathscr{V}\text{-mod} & \xrightarrow{\overline{T}} & \mathscr{V}\text{-mod} \\ {\scriptstyle (-)_\diamond}\uparrow & & \uparrow{\scriptstyle (-)_\diamond} \\ \mathscr{V}\text{-cat} & \xrightarrow[T]{} & \mathscr{V}\text{-cat} \end{array}$$

   commutes.

2. T *preserves exact squares.*

Relations in enriched setting

A module can be represented by a *cospan*

$$\mathscr{B} \xrightarrow{i_0} \mathrm{Coll}(R) \xleftarrow{i_1} \mathscr{A}$$

called the *collage* of R that becomes a two-sided discrete fibration in $(\mathscr{V}\text{-cat})^{op}$.
the category $\mathrm{Coll}(R)$ is defined as follows:

1. Objects of $\mathrm{Coll}(R)$ are the disjoint union of objects of $\mathscr{A}$ and $\mathscr{B}$.
2. $\mathrm{Coll}(R)(a, a') = \mathscr{A}(a, a')$ in case both a and a' are in $\mathscr{A}$.
3. $\mathrm{Coll}(R)(b, b') = \mathscr{B}(b, b')$ in case both b and b' are in $\mathscr{B}$.
4. $\mathrm{Coll}(R)(b, a) = R(b, a)$ in case b is in $\mathscr{B}$ and a is in $\mathscr{A}$.
5. $\mathrm{Coll}(R)(a, b) = \perp$ in case a is in $\mathscr{A}$ and b is in $\mathscr{B}$.

Examples of functors with BCC

The *Kripke-polynomial* 2-functors $T : \mathscr{V}\text{-cat} \to \mathscr{V}\text{-cat}$, given by the grammar

$$T ::= Id \mid \mathrm{const}_{\mathscr{X}} \mid T + T \mid T \times T \mid T \otimes T \mid T^{\partial} \mid \mathbb{L}T$$

The 2-functor T^{∂} (the *dual* of the 2-functor T) is defined as the following composite

$$\begin{array}{ccccccc} \mathscr{V}\text{-cat} & \xrightarrow{(-)^{op}} & \mathscr{V}\text{-cat}^{co} & \xrightarrow{T^{co}} & \mathscr{V}\text{-cat}^{co} & \xrightarrow{(-)^{op}} & \mathscr{V}\text{-cat} \\ \mathscr{A} & \longmapsto & \mathscr{A}^{op} & \longmapsto & T(\mathscr{A}^{op}) & \longmapsto & (T(\mathscr{A}^{op}))^{op} \end{array}$$

The 2-functor $\mathbb{L}$ sends $\mathscr{A}$ to $[\mathscr{A}^{op}, \mathscr{V}]$ and $f : \mathscr{A} \to \mathscr{B}$ is sent to the left Kan extension along f^{op}.
The 2-functor $\mathbb{U}$ is defined as $\mathbb{L}^{\partial}$. It sends $\mathscr{A}$ to $[\mathscr{A}, \mathscr{V}]^{op}$.

The powerset functor on preorders

A preorder $\mathscr{A}$ is mapped to the set of *all subsets of the carrier* of $\mathscr{A}$, ordered by the so-called Egli-Milner order

$$B \leq A \Leftrightarrow \begin{cases} & \forall b \in B \, . \, \exists a \in A \, . \, b \leq_{\mathscr{A}} a \\ \wedge & \\ & \forall a \in A \, . \, \exists b \in B \, . \, b \leq_{\mathscr{A}} a \end{cases}$$

The Pos-collapse of $\mathbb{P}$ is the convex powerspace functor, which provides the Kripke semantics for negation-free modal logic in the same way as the usual powerset provides the Kripke semantics for classical modal logic.

The powerset functor on $\mathscr{V}$-cat

The objects of $\mathbb{P}\mathscr{A}$ are arbitrary $\mathscr{V}$-subsets $\varphi : |\mathscr{A}| \to \mathscr{V}$ of $\mathscr{A}$. For any $\varphi, \psi : |\mathscr{A}| \to \mathscr{V}$ put

$$\mathbb{P}\mathscr{A}(\varphi, \psi) = [|\mathscr{A}|, \mathscr{V}](\varphi, \psi^{\downarrow}) \otimes [|\mathscr{A}|, \mathscr{V}](\psi, \varphi^{\uparrow})$$

or, in a detailed formula, by

$$\mathbb{P}\mathscr{A}(\varphi, \psi) = \bigwedge_a [\varphi(a), \bigvee_{a'} \psi(a') \otimes \mathscr{A}(a, a')] \otimes \bigwedge_{a'} [\psi(a'), \bigvee_a \varphi(a) \otimes \mathscr{A}(a, a')]$$

that can be perceived as the "Egli-Milner condition in the $\mathscr{V}$-setting".

The powerset functor on $\mathscr{V}$-cat

Given a $\mathscr{V}$-functor $f : \mathscr{A} \to \mathscr{B}$ and $\mathscr{V}$-subset $\varphi : |\mathscr{A}| \to \mathscr{V}$, define $\mathbb{P}f(\varphi) : |\mathscr{B}| \to \mathscr{V}$ by

$$b \mapsto \bigvee_a |\mathscr{B}|(fa, b) \otimes \varphi a.$$

In other words, $\mathbb{P}f(\varphi)$ is the value of a left Kan extension of φ along $|f| : |\mathscr{A}| \to |\mathscr{B}|$. In particular, the equality

$$[|B|, \mathscr{V}](\mathbb{P}f(\varphi), \psi) = [|\mathscr{A}|, \mathscr{V}](\varphi, \psi \cdot |f|)$$

holds for all $\varphi : |\mathscr{A}| \to \mathscr{V}$, $\psi : |\mathscr{B}| \to \mathscr{V}$.

Then $\mathbb{P} : \mathscr{V}\text{-cat} \to \mathscr{V}\text{-cat}$ is a 2-functor.

Coalgebras and bisimilarity in enriched setting

Definition

A T-coalgebra is a $\mathscr{V}$-functor $\xi : \mathscr{X} \to T\mathscr{X}$. Elements of $\mathscr{X}$ are called states and ξ is the transition structure. A coalgebra morphism from $\xi : \mathscr{X} \to T\mathscr{X}$ to $\xi' : \mathscr{X}' \to T\mathscr{X}'$ is $\mathscr{V}$-functor $f : \mathscr{X} \to \mathscr{X}'$ such that $\xi' \cdot f = Tf \cdot \xi$. The category of T-coalgebras is denoted by $\mathsf{Coalg}(T)$ and we write $U : \mathsf{Coalg}(T) \to \mathscr{V}\text{-cat}$ and $V : \mathscr{V}\text{-cat} \to \mathsf{Set}$ for the respective forgetful functors.

Definition

Bisimilarity, or behavioural equivalence, is the smallest equivalence relation on elements of coalgebras generated by pairs

$$(x, VUf(x))$$

where x is an element of a coalgebra and f is a coalgebra morphism.

∇ over $\mathscr{V}$-cat

The satisfaction relation should be a $\mathscr{V}$-module

$$\Vdash : \mathscr{X} \otimes \mathcal{L} \to \mathscr{V},$$

or equivalently,

$$\Vdash : \mathcal{L} \longrightarrow\!\!\!\!\!\!\!\!\shortmid\;\;\;\;\; \mathscr{X}^{op} .$$

Next we want to assume that $\mathcal{L}$ comes equipped with a ∇-operator.
The ^{op} makes it necessary to take formulas of the kind $\nabla\gamma$ not from $T\mathcal{L}$ but from $T^{\partial}\mathcal{L}$.
Recall that $T^{\partial}(\mathscr{X}) = (T(\mathscr{X}^{op}))^{op}$, so that T and T^{∂} agree on discrete $\mathscr{X}$. So we assume that we have an algebra

$$T^{\partial}\mathcal{L} \to \mathcal{L}$$

Semantics of ∇

Given a T-coalgebra ξ, we define the semantics of ∇ via the relation lifting of T^∂:

$$\Vdash(x, \nabla\gamma) = \overline{T^\partial}(\Vdash)(\xi(x), \gamma)$$

Notice that T^∂ preserves exact squares whenever T does.
In a diagram

$$\begin{array}{ccc}
T^\partial \mathcal{L} & \xrightarrow{\nabla_\diamond} & \mathcal{L} \\
{\scriptstyle \overline{T^\partial}(\Vdash)} \big\downarrow & & \big\downarrow {\scriptstyle \Vdash} \\
T^\partial(X^{op}) = (TX)^{op} & \xrightarrow{(\xi^{op})^\diamond} & X^{op}
\end{array}$$

Invariance under bisimulations

Proposition

If T preserves exact squares, then ∇ is invariant under bisimilarity.

Idea of the proof: To $\Vdash : \mathscr{X} \otimes \mathcal{L} \to \mathscr{V}$ corresponds a $\mathscr{V}$-functor $[\![\cdot]\!] : \mathcal{L} \to [\mathscr{X}, \mathscr{V}]$. The fact that all $\varphi \in \mathcal{L}$ are invariant under bisimilarity implies that

$$[\![\cdot]\!]_\xi : \mathcal{L} \to [U\xi, \mathscr{V}]$$

is natural in ξ.
We have to show that

$$[\![\nabla \cdot]\!]_\xi : T^\partial \mathcal{L} \to [U\xi, \mathscr{V}]$$

is also natural in ξ. We use the commutativity of:

$$\begin{array}{ccccc}
T^\partial \mathcal{L} & & \xrightarrow{\quad\nabla\quad} & & \mathcal{L} \\
{\scriptstyle T^\partial [\![\cdot]\!]} \downarrow & & & & \downarrow {\scriptstyle [\![\cdot]\!]} \\
T^\partial [\mathscr{X}, \mathscr{V}] & \xrightarrow{\delta_{\mathscr{X}^{op}}} & [T\mathscr{X}, \mathscr{V}] & \xrightarrow{[\xi, \mathscr{V}]} & [\mathscr{X}, \mathscr{V}]
\end{array}$$

Example: ∇ for $\mathbb{U}$-coalgebras

Recall that ∇ is an algebra for the functor $\mathbb{U}^\partial = \mathbb{L}$, hence

$$\nabla : \mathbb{L}(\mathcal{L}) \to \mathcal{L}.$$

We have
Given a $\mathbb{U}$-coalgebra $\xi : \mathscr{X} \to \mathbb{U}\mathscr{X}$ and $\gamma \in \mathbb{U}^\partial(\mathcal{L}) = \mathbb{L}(\mathcal{L})$ we have

$$\begin{aligned} \Vdash(x, \nabla\gamma) &= \overline{\mathbb{U}^\partial}(\Vdash)(\xi(x), \gamma) \\ &= \bigwedge_{y \in \mathscr{X}} [\xi(x)(y), \bigvee_{\varphi \in \mathcal{L}} \Vdash(y, \varphi) \otimes \gamma(\varphi)] \end{aligned}$$

For $\gamma = \mathcal{L}(-, \varphi)$ we obtain the semantics of $\Box$ from Bou et al.

$$\Vdash(x, \nabla\mathcal{L}(-, \varphi)) = \bigwedge_{y \in \mathscr{X}} [\xi(x)(y), \Vdash(y, \varphi)]$$

Example: ∇ for $\mathbb{L}$-coalgebras

Given a $\mathbb{L}$-coalgebra $\xi : \mathscr{X} \to \mathbb{L}\mathscr{X}$ and $\gamma \in \mathbb{L}^{\partial}(\mathcal{L}) = \mathbb{U}(\mathcal{L})$ we have

$$\begin{aligned} \Vdash(x, \nabla\gamma) &= \overline{\mathbb{L}^{\partial}}(\Vdash)(\xi(x), \gamma) \\ &= \bigwedge_{\varphi \in \mathcal{L}} [\gamma(\varphi), \bigvee_{y \in \mathscr{X}} \Vdash(y, \varphi) \otimes \xi(x)(y)] \end{aligned}$$

If $\gamma = \mathcal{L}(\varphi, -)$, we obtain the semantics of the $\Diamond$-operator from Bou et al.

$$\Vdash(x, \nabla\mathcal{L}(\varphi, -)) = \bigvee_{y \in \mathscr{X}} \Vdash(y, \varphi) \otimes \xi(x)(y).$$

Example: ∇ for $\mathbb{P}$-coalgebras

Consider a quantale $\mathscr{V}$ such that $\otimes = \wedge$. Given a $\mathbb{P}$-coalgebra $\xi : \mathscr{X} \to \mathbb{P}\mathscr{X}$, the ∇-semantics wrt ξ is given as follows. Observe that $\mathbb{P} = \mathbb{P}^{\partial}$, thus $\mathcal{L}$ is a $\mathbb{P}$-algebra. For every $x \in \mathscr{X}$ and $\gamma \in \mathbb{P}^{\partial}(\mathcal{L}) = \mathbb{P}(\mathcal{L})$ we have

$$\begin{aligned}
\Vdash(x, \nabla\gamma) &= \overline{\mathbb{P}^{\partial}}(\Vdash)(\xi(x), \gamma) \\
&= \bigwedge_{y \in \mathscr{X}} [\xi(x)(y), \bigvee_{\varphi \in \mathcal{L}} \Vdash(y, \varphi) \otimes \gamma(\varphi)] \\
&\otimes \bigwedge_{\varphi \in \mathcal{L}} [\gamma(\varphi), \bigvee_{y \in \mathscr{X}} \Vdash(y, \varphi) \otimes \xi(x)(y)]
\end{aligned}$$